Fostering Disciplinary Thinking and Academic Practice through Inquiry-based Learning in Mathematics

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Outline of Lecture

- What does it mean to learn mathematics as a discipline?
- How to teach students to think like a mathematician?
- Examples of inquiry-based learning: how to promote disciplinary thinking in our students?

Two Orientations to Math Learning

There are two orientations to mathematics learning that are relevant to the design of the syllabuses. They are:

1. Learning mathematics as a tool - This orientation places emphasis on using mathematics as a tool to solve problems. The selection of content is based on its value in contexts where students are likely to apply mathematics.

2. Learning mathematics as a discipline - This orientation places emphasis on understanding the nature of mathematics. The selection of content is influenced by its value to illustrate the thinking and practices of mathematicians.

Every syllabus has elements of both orientations, but the balance between the two differs. For example, the design of the N(T)-Level Mathematics syllabus is influenced by a stronger orientation towards learning mathematics as a tool, whereas the design of the Additional Mathematics syllabuses is influenced by a stronger orientation towards learning mathematics as a discipline.

Nature of Mathematics

Nature of Mathematics

Mathematics can be described as a study of the properties, relationships, operations, algorithms and applications of numbers and spaces at the very basic levels, and of abstract objects and concepts at the more advanced levels. Mathematical objects and concepts, and related knowledge and methods, are products of insight, logical reasoning and creative thinking, and are often inspired by problems that seek solutions. Abstractions are what make mathematics a powerful tool for solving problems. Mathematics provides within itself a language for representing and communicating the ideas and results of the discipline.

Four Recurring Themes in the study of mathematics:

- **P**roperties and Relationships (concepts)
- **O**perations and Algorithms (skills)
- **R**epresentations and Communications (processes)
- **A**bstractions and Applications (processes)

Four Recurring Themes in Study of Maths

Big Ideas in Mathematics

Big ideas express ideas that are central to mathematics. They appear in different topics and strands. There is a continuation of the ideas across levels. They bring coherence and show connections across different topics, strands and levels. The big ideas in mathematics could be about one or more themes, that is, it could be about *properties and relationships* of mathematical objects and concepts and the *operations and algorithms* involving these objects and concepts, or it could be about *abstraction and applications* alone. Understanding the big ideas brings one closer to appreciating the nature of mathematics.

Eight clusters of big ideas are listed in this syllabus. These are not meant to be authoritative or comprehensive. They relate to the four themes that cut across and connect concepts from the different content strands, and some big ideas extend across and connect more concepts than others. Each cluster of big ideas is represented by a label e.g. big ideas about Equivalence, big ideas about Proportionality, etc.

Eight Clusters of Big Ideas (for Sec)

- **If DN, PM me**
- If doing nothing, private message me
- Invariance, Functions, Diagrams, Notations, Proportionality, Measures, Models, Equivalence
- For **primary school**, only **6 clusters of big ideas**: exclude functions and models (model method is under diagrams)
- **IMPEND**

Disciplinary Thinking / Academic Practice

- What do mathematicians do in their **academic practice** and how do they think **(disciplinary thinking)**?
- **Pose** and solve mathematical problems to understand structures of certain mathematical objects
- E.g. square each digit of a positive integer and add to obtain a new number; repeat this process for the new number
- Example of an **open investigative task**: no specific questions or problems given in task
- General problem: Investigate to find any patterns

Types of Mathematical Tasks

- E.g. procedural tasks, word problems, problems in real-world contexts, mathematical modelling tasks, problem-solving tasks, (open) investigative tasks, etc.
- Example of **problem-solving task** (need to use some problem-solving strategies to solve): 16 people shook hand with each other once. How many handshakes were there?
- Question is given: closed goal; answer is usually closed, although may be able to extend to generalise
- **Investigative task: open goal** (search for any patterns; pose problems to solve) and **open answer**; involves both **problem posing and solving**

Investigation as a Process

- Need to distinguish investigation as a **process** from the **task**
- Investigation as a process involves four main processes:
  - **Specialising** (examine specific or special cases), e.g. let's try what happen if there are only 2, 3, 4, … people shaking hand?
  - **Conjecturing** (search for patterns and formulate conjectures)
  - **Justifying** (prove or refute conjecture)
  - **Generalising** (generalise pattern), e.g. can the pattern for 2, 3, 4, .. people shaking hand be generalised to 16 people? *n* people?
- Investigation **(inductive reasoning)** can be used to solve problem-solving tasks as well as open investigative tasks

Four Main Mathematical Processes

- Specialising, conjecturing, justifying and generalising are from Mason et al. (1985, 2010), which they applied to problem-solving tasks, but I borrowed them for investigation process
- **Mason, J., Burton, L., & Stacey, K. (1985).** *Thinking mathematically* (Rev. ed.). Wokingham, England: Addison-Wesley.
- Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Harlow, England: Pearson.

Inductive vs. Deductive Reasoning

- Another way to solve problem-solving tasks without using investigation is by using **deductive reasoning**
- E.g. for the handshake problem, can reason like this: every pair of people will give one handshake; there are $\binom{16}{2}$ different pairs of people, so there will be $\binom{16}{2}$ handshakes
- Problem-solving tasks can be solved by **investigation (inductive reasoning)** or deductive reasoning
- Usually, most people will start with investigation because they cannot straightaway think of solution

Tasks vs. Processes

- **Problem-solving task:** **solve problem (process)** by **process of investigation** (inductive reasoning) or **deductive reasoning**; don't really pose problems to solve at the start (but can pose further problems at the end or extend problem to generalise)
- **Investigative task:** decide to search for any pattern (implicit general problem posing) and **process is investigation**; can pose more specific problems at a later stage to solve: **solving these problems (process)** can be by **investigation (process)** or **deductive reasoning**

Some References

- Yeo, J. B. W., & Yeap, B. H. (2010). Characterising the cognitive **processes in mathematical investigation**. *International Journal for Mathematics Teaching and Learning*. Retrieved from http://www.cimt.org.uk/journal
- Yeo, J. B. W. (2017). Development of a framework to characterise the **openness** of mathematical tasks. *International Journal of Science and Mathematics Education, 15*, 175-191. doi:10.1007/s10763-015-9675-9

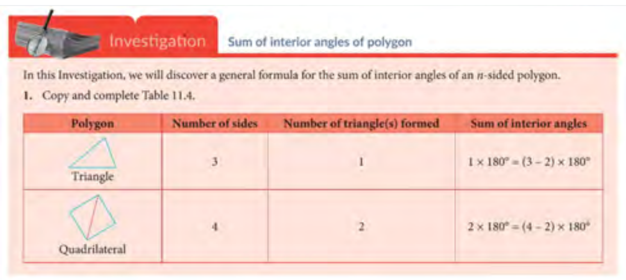
Disciplinary Thinking / Academic Practice

- **Cannot bring content** of academic practice of mathematicians down to primary or secondary level as it's beyond students' cognitive level
- But what we can do is to teach primary and secondary students how mathematicians think (bringing down mathematicians' **disciplinary thinking or processes** to students' level)
- Get students to **discover** mathematical concepts **at their level** by making them **think like mathematicians**
- But what mathematicians have discovered over a few millennia cannot be discovered by students on their own over 10-16 years of education

Disciplinary Thinking / Academic Practice

- Need to **guide** students to **discover** mathematical concepts
- How much guidance depends on readiness of students
- Guided-discovery is usual method in local secondary school textbooks (targeted at middle readiness students)
- Teacher's perspective: **guided-discovery learning** (Bruner, 1961)
- Students' perspective: **guided investigation**
- Unlike (open) investigative task, guided investigative task usually has a closed goal and answer

Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review, 31*, 21-32.


Investigation — Sum of interior angles of polygon

In this Investigation, we will discover a general formula for the sum of interior angles of an n-sided polygon.

1. Copy and complete Table 11.4.

Polygon	Number of sides	Number of triangle(s) formed	Sum of interior angles
Triangle	3	1	$1 \times 180° = (3 - 2) \times 180°$
Quadrilateral	4	2	$2 \times 180° = (4 - 2) \times 180°$

- Do most students tend to generalise from one or two examples?
- Even with more examples, how do you know pattern is correct? Is this even important?

Chords and Regions Worksheet

- No 3 chords intersect at same point inside circle: so that this will give the maximum number of regions
- Count number of distinct (not overlapping) regions
- 1, 2, 4, 8, 16, ___
- **Observed pattern** seems to be powers of 2: $T_n = 2^{n-1}$
- But this is **not actual / underlying pattern**
- Observed pattern is just a **conjecture** to be proven or refuted

Investigation Sum of interior angles of polygon

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- How do we know pattern is correct?
- How do we know that there will always be (n – 2) triangles?

Polygon	Number of sides	Number of triangle(s) formed	Sum of interior angles
Pentagon			
Hexagon			
Heptagon			

- Heptagon: n = 7 vertices; from one vertex, draw line to each of the 4 (= n – 3) vertices to obtain 5 (= n – 2) triangles (can be a bit abstract)
- Another pattern: as n increases by 1, we will get one more triangle

3. From Table 11.4, what is the general formula for the sum of interior angles of an n-sided polygon?
4. How can you tell that the general formula will always work for any n-sided polygon?
Hint: Consider the pentagon in Fig. 11.20. What happens if you add a point (represented by the cross) to make it into a hexagon? Will you add one more triangle? What if the point is somewhere else? Will you always add one more triangle when you change a pentagon into a hexagon?

Pentagon
Fig. 11.20

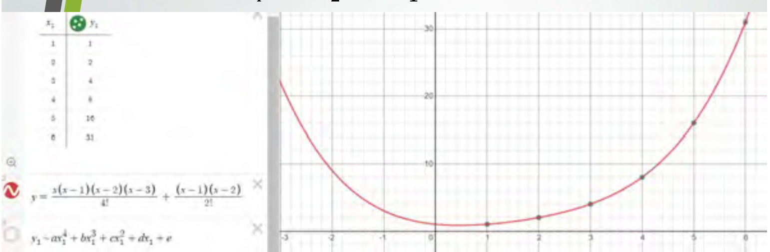
- When we add one more point to n-sided polygon to form (n + 1)-sided polygon, we will always get one more triangle
- Only convex polygon in syllabus
- Formal proof may be too difficult or not possible: **reason to believe that pattern will continue** may be good enough
- For some guided investigation (e.g. chords and regions worksheet), it is hard to find such reason

Chords and Regions Worksheet

- 1, 2, 4, 8, 16, 31
- $T_n = \binom{n}{4} + \binom{n-1}{2} + \binom{n}{1}$: polynomial of degree 4
- 1, 2, 4, 8, 16, ___
- **The next term can be any number!**
- E.g. 1, 2, 4, 8, 16, 7: unique polynomial of degree at most 5 (see https://en.wikipedia.org/wiki/Lagrange_polynomial)
- If there is **no context**, given 1, 2, 4, 8, 16, ___, we will expect a **simple pattern** from students, namely, 32
- https://www.desmos.com/calculator/w7w9edcjay

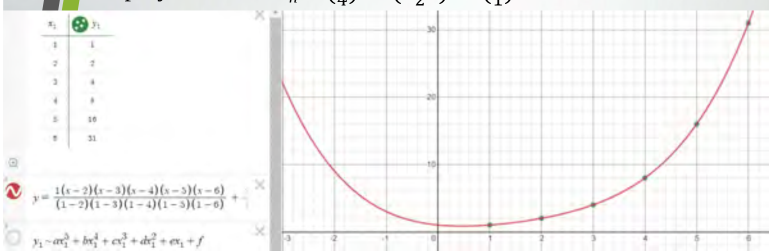
1, 2, 4, 8, 16, 31

- Using $T_n = \binom{n}{4} + \binom{n-1}{2} + \binom{n}{1}$



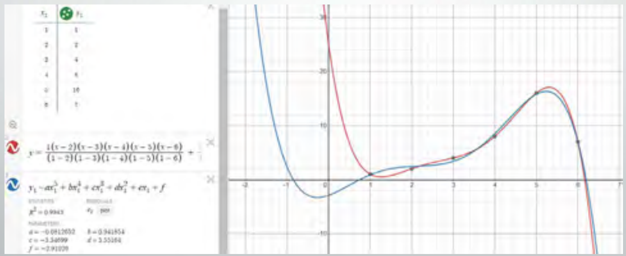
1, 2, 4, 8, 16, 31

- Using Lagrange polynomial which will reduce to same polynomial as $T_n = \binom{n}{4} + \binom{n-1}{2} + \binom{n}{1}$



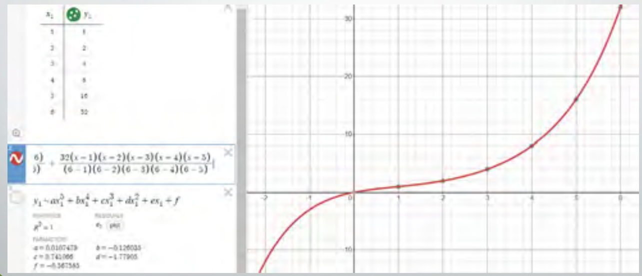
1, 2, 4, 8, 16, 7

- Lagrange polynomial (red curve) more accurate than curve of best fit using least squares method (blue curve)



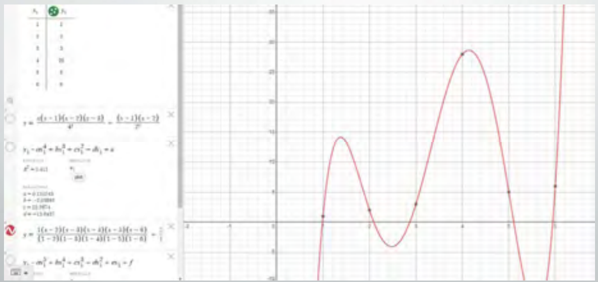
1, 2, 4, 8, 16, 32

- Can still use Lagrange polynomial instead of powers of 2



1, 2, 3, 28, 5, 6

- Almost a straight line except for one point



Number Patterns (Primary and Sec 1)

- 1, 2, 4, 8, 16, … What is the next term?
- If there is **no context**, given 1, 2, 4, 8, 16, ___, we will expect a **simple pattern** from students, namely, 32
- If students argue why next term is 31 with $T_n = \binom{n}{4} + \binom{n-1}{2} + \binom{n}{1}$, you mark them correct
- Don't give only two terms, e.g. 1, 2, … as there are now so many possibilities for next term
- **Give at least 4 or 5 terms** for students to see 'obvious' pattern

Pattern that is not always true (for Pri)

- Students learn multiples in primary schools
- For multiples of 3, e.g. 27, if you add the digits, the sum 9 is also a multiple of 3 (divisibility test for 3)
- Is this also true for multiples of other whole numbers?
- For multiples of 9, e.g. 783, if you add the digits, the sum 18 is also a multiple of 9 (divisibility test for 9)
- So, is this always true?
- Counter example: 12 is a multiple of 2 but sum of digits, 3, is not multiple of 2

Key Takeaway #1: Processes of Investigation

- Observed pattern from **specialising** in investigation (induction) is only a **conjecture** to be proven or refuted (but mathematical induction previously taught in JC is rigorous proof: P_n is true $\Rightarrow P_{n+1}$ is true; then P_1 is true $\Rightarrow P_2$ is true etc.)
- To refute a conjecture: just need a counter example
- To prove or **justify** a conjecture: formal proof (e.g. mathematical induction) or some reason to believe that pattern will continue
- If conjecture is justified, **generalisation** has occurred: observed pattern is underlying general pattern
- Four main mathematical processes from Mason et al. (1985, 2010)

Guided Investigation

- Guided investigation / guided-discovery learning is **not really what mathematicians do**, but it may be good enough for students to have an idea of what the **process of investigation** is
- To give students a foretaste of what academic practice (or disciplinary thinking) is, we can make the task **more open**
- Why would people want to find the sum of interior angles of polygon in the first place?
- E.g. to construct a regular pentagon structure or to make a drawing first, need to find size of each interior angle

Regular Hexagonal Table?

- It was meant to be a regular hexagon, but it did not turn out this way
- No one told designer or manufacturer to take care of the size of each interior angle

Teaching through Problem Solving

- Suppose you want to custom make a table in the shape of a regular hexagon like this
- You need to tell the manufacturer the size of each interior angle of the hexagon
- Find the size of each angle

Teaching through Problem Solving

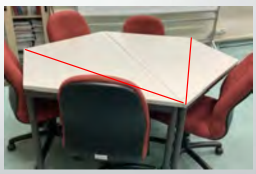
- Problem must be given **before** students learn formula for finding sum of interior angles of polygon (or else it will just be a simple application question)
- During seatwork, do not tell students the solution but ask guiding questions to help students think (focus is on thinking **processes**, not getting the answer per se), e.g.
  - What have you learnt that can help you find the size of each angle?
  - Can you divide the hexagon into triangles? How will this help?

Teaching through Problem Solving

- After students have obtained solution, what else can you do?
- **Train students to pose problems** to solve:
  - Is there an **alternative method** to solve this? Which method is more efficient?
  - **What else can I investigate?** Can I generalise? Can I extend problem?
  - How do I know the general formula always work? Is there a reason to believe that it will always work? Can I prove it?
- These are what mathematicians would usually do: find more efficient method, or general formula and proving that it works

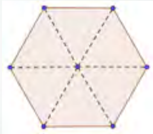
Teaching through Problem Solving

- More specific questions to ask in this case:
  - Can I generalise formula for finding sum of interior angles for regular hexagon to irregular hexagon? To any polygon?
- If students divide regular hexagon into 4 triangles like this, can generalise to irregular hexagon any polygon

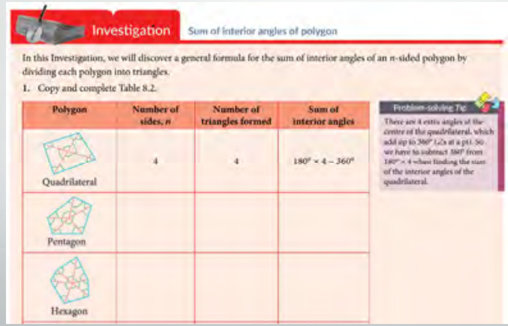


Teaching through Problem Solving

- But regular hexagon is special case: can divide into 6 equilateral triangles
- Then how to generalise?
- Sum of interior angles = $n \times 180° - 360°$
- Low readiness students have difficulty linking n-sided polygon to $(n - 2)$ triangles
- Easier for them to link n-sided polygon to n triangles and then subtract 360° in centre



Different Version in N(A) Textbook



Investigation Sum of interior angles of polygon

In this Investigation, we will discover a general formula for the sum of interior angles of an n-sided polygon by dividing each polygon into triangles.

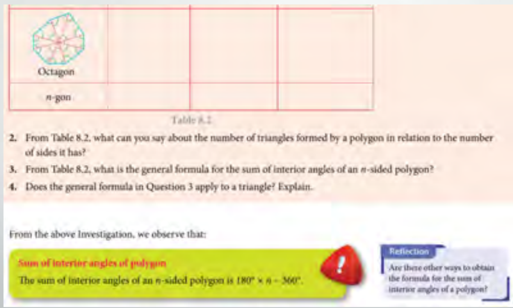
1. Copy and complete Table 8.2.

Polygon	Number of sides, n	Number of triangles formed	Sum of interior angles
Quadrilateral	4	4	$180° \times 4 - 360°$
Pentagon			
Hexagon			

Problem-solving Tip

There are 4 extra angles at the centre of the quadrilateral, which add up to 360° (∠s at a pt). So we have to subtract 360° from 180° × 4 when finding the sum of the interior angles of the quadrilateral.

Different Version in N(A) Textbook



Octagon			
n-gon			

Table 8.2

2. From Table 8.2, what can you say about the number of triangles formed by a polygon in relation to the number of sides it has?
3. From Table 8.2, what is the general formula for the sum of interior angles of an n-sided polygon?
4. Does the general formula in Question 3 apply to a triangle? Explain.

From the above Investigation, we observe that:

Sum of interior angles of polygon

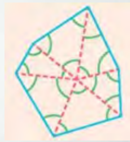
The sum of interior angles of an n-sided polygon is $180° \times n - 360°$.

Reflection

Are there other ways to obtain the formula for the sum of interior angles of a polygon?

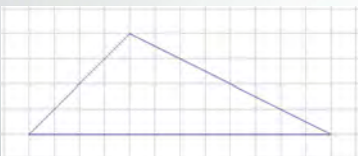
Teaching through Problem Solving

- Are the two methods related?
- https://www.geogebra.org/classic/vu69gact
- If you move middle point in top diagram to vertex on bottom left, it will look like bottom diagram



Teaching through Problem Solving (Pri)

- Find the area of a triangle with a base of 12 units and a height of 4 units. (Each small square in the square grid has a length of 1 unit.)



Question: How can we help students to think like a mathematician?

- Secondary school teachers can think of a trapezium instead

Teaching through Problem Solving

- **Choice of problem** is crucial when teaching through problem solving
  - Problem **rich enough** to allow multiple methods of solution to facilitate discussion **(not too closed)**
  - Problem **suitable to achieve specific goal** of guiding students to discover certain concepts **(not too open)**

Open Investigative Task

- Given a polygon, investigate.
- Very open, but this is **what mathematicians do**
- When ancient people looked at a polygon, what did they want to find?
  - What is the perimeter or area of the polygon?
  - What is the size of an interior angle of a regular polygon?
  - What are some properties of special quadrilaterals (4 sided polygons) such as square, rectangle, parallelogram, rhombus, trapezium and kite?
  - For a right-angled triangle, are there any special properties?
- **Too open**: need to scale down, e.g. find size of interior angle of regular polygon

Open Investigative Task

- Square each digit of a positive integer and add to obtain a new number; repeat this process for the new number
- Investigating any positive numbers is too open: above open investigative task is already scaled down to this specific process
- This kind of questions usually not in syllabus but good to stretch high readiness students to learn how to pose their own problems to solve

Inquiry-based Learning (IBL)

- Guided discovery learning, teaching through problem solving, or open mathematical investigation are examples of inquiry-based learning
- **Inquiry:** act of asking for information, or process of asking a question
- **Inquiry-based learning:** Get students to learn by **asking questions** (and finding the answers to the questions themselves)
- What mathematicians have discovered over a few millennia cannot be discovered by students on their own over 10-16 years of education
- **Guide** students to **discover** mathematical concepts
- How much guidance depends on readiness of students

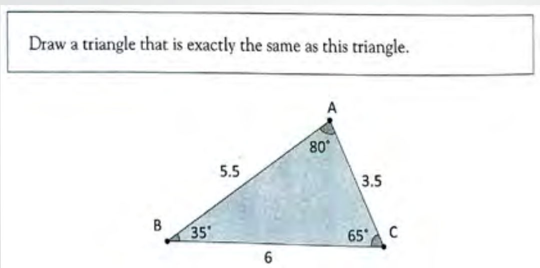
Inquiry-based Learning (IBL)

less open ← continuum → more open

Guided-Discovery Learning	Teaching through problem solving	Open Investigation
• **Guided investigation** (process) • More guided / less open	• **Problem solving** via investigation (usually) or deductive reasoning • **Pose more problems** after solving given problem • Less guided / more open	• **Pose any problems to solve** • Problem solving via investigation or deductive reasoning • Even more open

* Different forms of **Inquiry-based Learning**

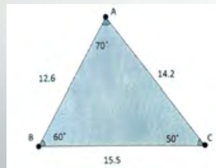
Teaching through Problem Solving

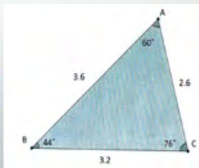


Takahashi, A. (2021). *Teaching mathematics through problem-solving: A pedagogical approach from Japan.* New York: Routledge.

Choice of Triangle

- **Cannot anyhow choose dimensions of triangle**, e.g. if you decide on the three lengths of triangle, the angles are fixed already because of SSS congruent test (use cosine rule to find first angle, sine rule to find second angle, etc.)
- Other popular triangles among Japanese teachers





Choice of Triangle

The success of this lesson depends on what triangle the teacher gives students to recreate. Research has shown that there are four essential criteria:

- The triangle itself is small enough to fit in the pages of the students' notebooks.
- The triangle is big enough to be seen during the whole class discussion.
- The length of each side can be measured and made using the students' rulers; they must be rounded to the nearest tenth.
- Each angle should be a whole number so students can use their protractors.

Congruent Triangle

- Most students will realise that they do not need all the six measurements to draw the same (or congruent) triangle
- Train students to **ask or pose following questions themselves**:
  - **Why is this so?**
  - **What can I do next?** [general problem]
  - How is my solution and my classmates' solutions same or different? Why?
  - [May lead to these questions] What is the least number of measurements do I need to draw the same triangle? And which are these measurements?

Possible Students' Solutions

SSS (a) The lengths of all three sides, AB, BC, and AC.

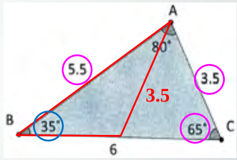
SAS (b) The lengths of two sides, AB and BC, and the angle in between ∠ABC.
(c) The lengths of two sides, BC and CA, and the angle in between ∠ACB.
(d) The lengths of two sides, AB and AC, and the angle in between ∠BAC.

AAS (e) The length of side AB, and two angles, ∠ABC and ∠BAC.
(f) The length of side BC, and two angles, ∠ABC and ∠ACB.
(g) The length of side AC, and two angles, ∠BAC and ∠ACB.

Others? (h) Three measurements are needed that are different than the ones listed above.
(i) Four, five, or six measurements are needed.

Takahashi, A. (2021). *Teaching mathematics through problem-solving: A pedagogical approach from Japan.* New York: Routledge.

Possible Students' Solutions



- Is minimal no. of measurements needed = 3?
- It depends which 3, e.g. **AAA cannot work**
- What if students use non-included ∠*ACB* = 65°, *AB* = 5.5 cm and *AC* = 3.5 cm?
- **It works!** SSA works?!
- In general, SSA is **not** a congruence test, but there are **exceptions**:
  - Given non-included angle is acute (in this case, ∠*ACB*) and side opposite this angle (in this case, *AB*) is longer than the other given side (in this case, *AC*)
- But **SSA will not work** if we use non-included ∠*ABC* = 35°, *AB* = 5.5 cm and *AC* = 3.5 cm: we can draw red triangle

Congruence Tests for Triangle

- Three main congruent tests learnt from previous problem:
  - SSS
  - SAS (included angle)
  - AAS (or ASA): does not matter whether side is included or not
- RHS: special case of SSA (angle = 90°) that is a congruence test
- **In general, SSA is not a congruence test unless:**
  - given non-included angle is 90° (RHS),
  - given non-included angle is acute and side opposite this angle is longer than the other given side (see previous slide; not in syllabus),
  - given non-included angle is obtuse (not in syllabus), **etc.**

Congruence Tests for Triangle

- Can students conclude from the previous problem that the congruence tests for a triangle is SSS, SAS and AAS?
  - Do you want students to **conclude from one example?** Have all representative types of triangles been covered? What about right-angled or obtuse-angled Δ? (recall chords and regions worksheet)
- **Problem is too big** in the sense that students learn so many congruent tests at one go. Can they cope?

Congruence Tests for Triangle

- Middle readiness students: **Need to consolidate** after this problem by looking at each congruence test in detail: to show that it really works for all triangles (see **guided investigation** in textbooks), and to practise proving after teaching each congruence test
- High readiness students: There are always middle readiness students in a class of mostly high readiness students, so above consolidation will be helpful
- Low readiness students: Maybe just use guided investigation in textbooks directly
- How open or guided depends on **readiness of students**

Guided Investigation in Textbook

Investigation SAS Congruence Test

Part 1

1. Try to construct $\triangle XYZ$ such that $XY = 3$ cm, $YZ = 6$ cm and $X\hat{Y}Z = 50°$ in as many ways as possible. Compare the triangle you have drawn with those drawn by your classmates.
2. Do you get the following triangles in Fig. 9.12 (not drawn to scale)? Both triangles are congruent to each other, which means that you can map both triangles together by a reflection. Is it possible to get other triangles?

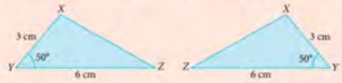


Fig. 9.12

3. Try to construct $\triangle XYZ$ for other dimensions, where XY and YZ have a fixed length and $X\hat{Y}Z$ is a fixed angle, in as many ways as possible and see if you always get a unique triangle (regardless of orientation).
4. Notice that the given angle is between the two given sides XY and YZ. Therefore, $X\hat{Y}Z$ is called the **included angle**.
5. What can you conclude from **Part 1** of this Investigation?

Guided Investigation in Textbook

Part 2

6. Try to construct $\triangle ABC$ such that $AB = 5$ cm, $AC = 3$ cm and $A\hat{B}C = 30°$ in as many ways as possible.
7. Fig. 9.13 (not drawn to scale) shows one possible $\triangle ABC$ that satisfies the given dimensions.

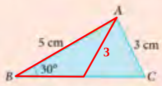


Fig. 9.13

Is it possible to construct a different $\triangle ABC$ (excluding a laterally inverted triangle)?

8. Notice that the given angle is not in between the two given sides AB and AC, i.e. $A\hat{B}C$ is *not the included angle*.
9. What can you conclude from **Part 2** of this Investigation?

Key Takeaway #2: Inquiry-based Learning

- Mathematicians either solve problems posed by others, or they find or pose their own problems to solve
- **Teaching through problem solving** and open investigation are **more aligned** to what mathematicians do, but open investigation is too open, while teaching through problem solving may or may not be too open depending on task (crucial to choose good task)
- **Guided investigation / guided-discovery learning** is not really what mathematicians do, but it may be good enough for students to think like mathematicians in terms of learning the **four main mathematical processes** (Mason et al., 1985, 2010); what is lacking is problem posing

Conclusion

- Learning mathematics as a discipline means doing mathematics as mathematicians would do in their academic practice, i.e. learning to **think like mathematicians**
- Use of **inquiry-based learning** to help students engage in what mathematicians do: four **processes** of investigation and problem solving (specialising, conjecturing, justifying and generalising), and problem posing
- Is learning mathematics about drilling and passing exams, or learning to solve unfamiliar problems by thinking like a mathematician? **Will you help to convince** your maths colleagues in your school to teach students to think like a mathematician?

New Article

- Special issue of The Mathematician Educator (2022 Volume 3 Issue 1)
- https://ame.org.sg/tme

Motivating Mathematics Students and Cultivating the Joy of Learning Mathematics

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The underlying basis of the self-determination theory (SDT) is that people are inherently motivated to learn if their basic psychological needs of autonomy, competence and relatedness are met. The theory also provides a comprehensive taxonomy on the different types of extrinsic and intrinsic motivations. For example, identified and integrated extrinsic motivations are based on a sense of *value* while intrinsic motivation is based on *interest*. In this article, I will put the theory into practice, suggesting in more concrete terms how teachers could motivate their students to learn mathematics. First, I will describe some applications within mathematics and in the real world which could be used to motivate students extrinsically by helping them see the value of what they are studying. Moreover, real life examples might also help students *relate* mathematics to their own experiences. Then I will provide some examples of catchy mathematics songs and amusing videos which could be used to motivate students intrinsically by arousing their interest. I will also discuss how to build up students' *competence* in mathematics by developing concepts using examples and not definitions, and by using guided discovery learning and guided proofs, which could also provide *autonomy* support for the students. I will examine how the practice of procedural skills could be structured more effectively and how mathematics puzzles and gamification could make such practice more enjoyable. Lastly, I will draw on a research study to inform what Singapore teachers are doing to motivate their students.