ASSOCIATION OF MATHEMATICS EDUCATORS

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Maths Buzz

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President's Message...

Dear AME Members,

The Association of Mathematics Educators celebrated its 10th Anniversary on 29th May 2004. Under the able leadership of the past 5 executive committees the Association has gained not only national but also international recognition. Our publications, Maths Buzz and The Mathematics Educator have both served their purposes well. The Maths Buzz has helped to wet the appetite of our members – kept them posted briefly about upcoming events and shared with them some interesting snippets on mathematical ideas or teaching episodes. The Mathematics Educator has been very successful in publishing research papers related to the teaching and learning of mathematics. It is currently a much sought after publication by international researchers. This is certainly a significant milestone for the Association. To all our past and present editors of Maths Buzz and The Mathematics Educator we say, Thank You and Well Done!

The Association has in the last decade provided numerous professional development and enrichment activities for members and mathematics teachers. It has co-organized conferences with Educational Research Association of Singapore (ERA-AME-AMIC Conference 1999) and Mathematics & Mathematics Education AG at NIE (ICMI – EARCOME 2002). The Association has also contributed towards making Mathematics a fun-filled activity for pupils. A few thousand pupils have already done Mathematics Trails produced by the Association. To celebrate the 10th Anniversary, the Association produced a student journal book which pupils may use for their mathematics journal entries. Several hundreds of copies have already been purchased by mathematics teachers for their pupils.

At the 11th annual general meeting of the Association, I was elected the President of the Association. This is my second time as President of the Association. The sixth executive committee of the Association will strive to do their best they can for the Association. A five-day work week coupled with exciting initiatives such as Teach Less, Learn More and Modes of Alternative Assessment are currently the challenges the Association is addressing. TIMSS (Trends in Mathematics and Science Study) 2003 results will be released in December 2004. This time round our Primary Four and Secondary Two pupils participated in the Study and we are looking forward to the results.

We need your support and cooperation for the continued well-being of the Association and to scale it to greater heights. We value your views and feedback. Please feel free to e-mail us about any matter related to the Association. The Association's homepage is http://math.nie.edu.sg/ame/ .

Berinderjeet Kaur

President, AME (2004 – 2006)

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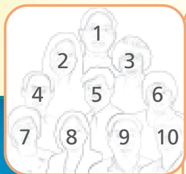
Members of the Executive Committee (2004 - 2006)



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Opening Textbook Sums for Mathematical Creativity

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Introduction

Teachers in the primary schools are very conversant with the standard type of problem sums that are set for pupils to work individually, after teaching a topic and equipping them with the necessary basic concepts and procedures to apply. Very often these tasks are termed as *one-step, two-step or multiple-step problems* where pupils are taught certain procedures. For more complex questions that require analysis of part-whole or proportional relationships, model drawing is often taught as the only strategy to solve such type of problems. Examples of such challenging questions are shown in Figure 1.

1. **3/5 of Pr 6A and 3/4 of P 6B are girls. Both classes have the same number of girls and P 6A has 8 more boys than P 6B. How many pupils are there in P 6A?**
2. **At first, the ratio of Susan's money to Mary's money was 5 : 3. After Susan gave $20 to Mary, they each had an equal amount of money. How much money did Mary have at first?**
3. **25% of a class of 32 pupils are boys. If some more boys join the class, the percentage of boys increases to 40%. How many boys join the class?**

Figure 1: Examples of Challenge Sums

There is however an over reliance on such type of challenge problems in the Singapore syllabus to assess so called higher–order analytical thinking skills of the pupils. This also led to an undesirable practice of over emphasizing a particular method like *model* drawing for children to solve similar structured word problems across related arithmetic topics like whole numbers, fractions, ratio and percent. In recent years, with the emerging trends towards emphasis *of process and thinking skills* in mathematics, teachers have been encouraged to teach problem solving heuristics using non-routine problems that would require strategies such as *guess and check; look for patterns; make supposition; work backwards etc. etc.* Very often these non-routine mathematical problems are used in supplementary lessons as enrichment for the better ability pupils where the emphasis is on the practice of heuristics on similar kind of questions.

Currently, most mathematics classrooms in Singapore are practicing the traditional approach of whole class expository teaching followed by pupil practice of routine exercises and regular written tests consisting of multiple-choice questions, short-answer and long-answer open response questions (Chang, Kaur, Koay and Lee, 2001). There is a need to equip teachers with a greater variety of mathematical tasks for problem solving that can enhance their teaching methods. At the same time students must encounter rich problems where they can offer evidence of their reasoning and find connections across topics to develop their mathematical creativity.

Mathematical Creativity

Creativity is not usually associated with the traditional image of school mathematics which is often seen as a static and well-defined body of knowledge. This portrayal of school mathematics has led to lessons where students tediously learn a collection of techniques by following predetermined rules. Hence it might be difficult to understand or even attempt to define what *mathematical creativity* is about for the learners in this context. Students should not only know the concepts and procedures but also how mathematics is created and used to learn new concepts and to solve problems in original and as well in as many ways possible. For mathematical creativity to develop, students must be confronted with rich tasks or problems. Carpenter and Lehrer (1999) propose that classrooms need to provide students with tasks that can activate in them the following interrelated mental activity for understanding and creative thinking to emerge: (a) develop appropriate relationships and seeking patterns, (b) extend and apply their mathematical knowledge, (c) reflect about their own experiences, (d) articulate what they know, and (e) make mathematical knowledge their own. The teaching methods for these thinking activities would include active participation of students on a variety of strategies and paying attention to the social aspect of learning. The creative teacher in such classroom moves away from the image as an expert who knows all and instead spurs the students to see themselves as autonomous learners who can think critically and make decisions.

Opening Textbook Sums for Mathematical Creativity

The author has conducted workshops on the use of open-ended tasks for many primary teachers in Singapore. For the purpose of this paper, we will focus on short open-ended questions that teacher can transform from standard questions commonly found in textbook exercises. Such questions can be integrated with normal lessons and do not require an extensive length of several periods or weeks for pupils to do. Teachers can use short open-ended problems for its role in developing deeper understanding of mathematical ideas, generating creative thinking and communication in students. Caroll (1999) found that short open-ended questions provided teachers with quick checks into students' thinking and conceptual understanding. They were no more time-consuming to correct than the worksheet questions that teachers normally give. When used regularly, the pupils in the study developed the skills of reasoning and communication in words, diagrams, or picture.

At the workshops, teachers were given opportunity to create open-ended problems and then they had to trial their problems on their own pupils in school. The characteristic features of an open-ended question are that there are many possible answers: it can be solved in different ways and if possible, accessible to pupils in mixed abilities classrooms. It should offer pupils room for own decision making and natural mathematical way of thinking. There are three categories of open-ended questions that teachers can create from textbook sums:

- **Problem with missing data or hidden assumptions**
- **Problem to explain a concept, procedure or error**
- **Problem Posing**

Teachers can create question with missing data or hidden assumptions from a textbook sum. Consider a routine sum that one may find in a primary textbook, figure 2. Formulated in such a closed structure, the teacher and pupils would have in mind the expected standard response of seeing it as a multiplication sum. However, in the same context of a Polar bear, teachers in the Netherlands (Van den Heuvel-Panhuizen, 1996) posed an open-ended situation for the pupils to solve, see Figure 3.

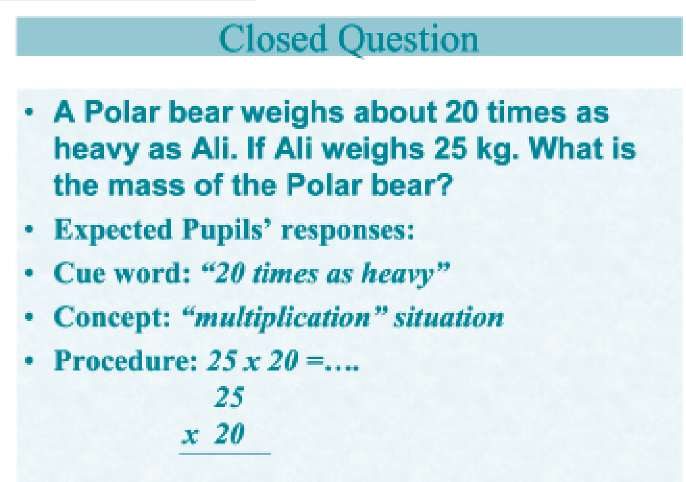


Figure 2: Closed questions with an expected standard response.

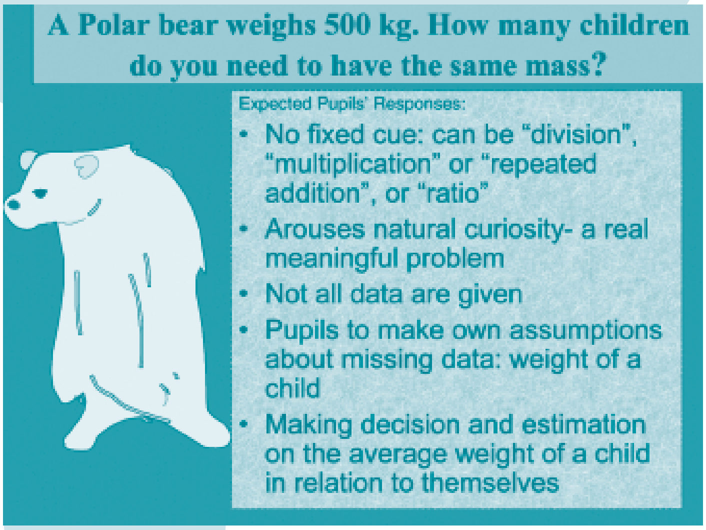


Figure 3: Open-ended situation with a missing information

According to Van den Heuvel-Panhuizen, the Polar Bear problem (Figure 3) represents an important goal of mathematics education. In an open-ended situation, pupils in addition to applying calculation procedures are also required to solve realistic problems where there is no known solution beforehand and not all data is given. It would require pupils' own contributions, such as making assumptions on the missing data. Without giving the children's weight, it becomes a real problem and the pupils have to think about and estimate the weight of an average child. There is no cue word for students to figure which operation to use as in the closed question, Figure 2.

A routine textbook sum is normally structured with all the necessary elements for pupils to find an unknown. Using this sum, a teacher can leave out some of the given conditions whereby students have to make realistic assumptions and decisions that would require their reasoning in order to find solutions as in the above Polar bear example. Another example, at the upper primary level for the topic on "average", the standard sums are usually set with a given quantity of measures for pupils to find the average. In an open-ended situation, they can transform the question to give pupils the known average so they would have to find the unknown elements that can have many answers as shown in this question:

"You have five different brands of the same size bottles of mineral water. The average cost for a bottle is $1.55. What might be the actual prices of each of the five bottles of water?"

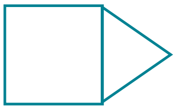
Creating open-ended questions that ask students to explain a concept or error can be easily adapted from textbook sums. Instead of asking for the answer to the textbook questions add for example "*Explain to a friend who does not understand how you arrived at the answer, you may using diagrams or manipulatives to help you.*" Or a wrong solution can be posed to pupils that requires them to analyse the errors and then explain how the problem be solved correctly. Below is an example:

Textbook sum: **Round off 5.37 to the nearest tenth.**
Convert to open-ended question by adding this statement to the sum:

Maria is helping her brother to do this sum. He wants to know why you change the 3 to 4 and drop the 7. How can Maria explain this to her brother?

The third category of open-ended questions is *problem posing*. This kind of task is gaining popularity in many schools where students are given opportunities to construct questions based mathematical concepts or skills that they have learned. For the younger students they can write number stories when given a number statement like "5 + 6 = 11". For teachers to create *problem posing* questions they can take a textbook sum, leave out the question and then ask students to pose whatever question which they think would fit the given problem situation. For example:

The figure is made up of a square and an equilateral triangle of side 5 cm.



What is the question?...
Solve it:

Or the teacher can pose a mathematical statement and ask students to make up a word problem, for example:

Write a number story for this sum: 1/3 ÷ 3 = ?

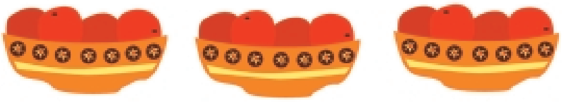
Teachers' Sharing and their Students' Work

In this section, we share the teachers' experience in using open-ended questions with their students. At the workshops, the teachers found little difficulty in adapting the questions but initially many of them were apprehensive about giving their students such tasks. It has never been part of their teaching where they require students to give explanations and reasons for their solutions. The practice has always been students are given problems that have only one answer and one taught method of finding it. When the teachers had trialed the open-ended questions with their students, many of them were surprised by the rich responses that most of their students could give. Of course, there were also reports by some teachers that many students seem to lack reasoning and communication skills initially. The following are experiences of three teachers who had used open-ended questions and their students' responses:

The Bowls of Oranges

Miss A, a grade four teacher adapted this open-ended question, figure 4 as an extension to the routine problem sums that she often gave her pupils. The open-ended version demands higher order thinking in pupils.

Open-ended Version



Bowls of Oranges

There are 12 oranges to put into bowls. Each bowl must have the same number of oranges. Show how you could put the oranges into bowls.

Cognitive demands of the open-ended task

- Pupils to make own assumptions about the missing data: the number of bowls/number of oranges in each bowl
- Pupils to access relevant knowledge multiplication, division, fraction, factors etc..
- Pupils to display number sense & equal grouping patterns
- Pupils to use the strategy of systematic listing
- Pupils to communicate their reasoning using multiple modes of representation
- Pupils to display creativity in as many possible strategies and solutions

Figure 4: Open-ended task with higher cognitive demands

A standard closed textbook question opposite to this open-ended task would normally be of this format:

> There are 12 oranges to put into 3 bowls. Each bowl must have the same numbers of oranges. How many oranges are there in each bowl?

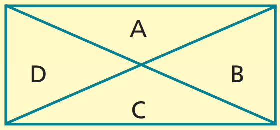
Miss A observed that the pupils were extremely excited over the open-ended question when told that they were to work in small groups as *Math Investigators* to solve this problem. They wanted to do better than other groups in finding more solutions. The pupils were thrilled by the idea of being *Math Investigators* as reflected in their eagerness to start the project. They liked the investigative part and the freedom to decide on how to present their solutions. Initially some pupils had difficulty understanding the statement in the problem: "each bowl must have the same number of oranges" and needed more clarification from the teacher.

To her surprise, most of her pupils were able to respond appropriately and the variety of the children's thinking was enlightening. Some children could reflect upon analysing the question, that it cannot be solved straightaway as there was not enough given information. Many groups were able to give multiple representations and made connections between repeated addition, multiple and division. They drew pictures accompanied by mathematical sentences using appropriate symbols. They were also systematic in their listing of possible answers. The teacher observed that among the above average pupils some were quick to catch onto the idea of listing the factors of 12 to get 6 combinations. This was related to a new topic under "factors and multiples" that they had just learned at grade 4. One pupil exclaimed during discussion "*this made understanding factors so easy*!"

The Rectangle Problem

The rectangle problem below was used by Mrs. B with her grade six class.

> A rectangular piece of paper is cut into four pieces, A, B, C and D as shown in the figure. The piece A is given to Alice and the piece B is given to Bessie. Do you think Alice has a bigger piece of paper than Bessie? Explain your answer in at least two ways.



The task was open-ended to allow pupils to apply non-algorithm thinking; to access relevant knowledge on properties of rectangles and triangles; to apply geometric concepts and visualization skills to manipulate or dissect a shape; they might make assumptions about the missing data which they think are necessary and to display their creativity in using a variety of strategies for the solution. The pupils had prerequisite knowledge of properties of rectangle and triangle and they had learnt the area formula for both shapes. However this is the first time that pupils were exposed to such open-ended problem.

When the discussion started, they seemed a little handicapped as it was also the first time they did group activity for math. Most pupils seemed confident in handling the preliminary activity which is a closed problem but when they received this open-ended version they were hesitant and were doubtful that they could perform the task. Their most common query was whether there was missing information that the teacher forgot to give in the worksheet. They questioned whether they can solve the problem without given data for the length and breadth. The pupils felt insecure about handling a mathematical problem without any numbers involved. It was only after much assurance that they were convinced that the problem was solvable. When they were reminded of the preliminary activity, most groups started using the cutting method but some had difficulty orientating the pieces. Surprisingly those pupils who were normally not proficient in math actually got the correct answer first but they had problem verbalising their methods. Teacher pressed for justifications no matter what answer the pupils decided. With the teacher's guidance and hints, listed below were four good methods extacted from the solutions of the pupils:

- **Cutting and matching A to B**
- **Cutting into 8 equal triangles**
- **By formula, assuming some dimensions**
- **Drawing unit squares.**

There were two groups of pupils who intuitively perceived that they are the same by observation. There was also a group who use abstract algebraic reasoning to justify their conclusion.

Teachers' Reflection

This approach of opening the textbook sums for pupils to do has enabled teachers to see pupil's thinking rather than the teacher's own thinking through closed questions that have predetermined method and answer. To encourage others to implement open-ended questions in their teaching, excerpts of two of the teachers, Miss A and Mrs. B's reflection of their experience are expressed in the following:

Miss A: (Bowls of Oranges)

"As we become more and more exam-oriented, activities are relegated to the end of the priority need. Yet, in doing so, we are effectively putting blinders on the pupils such that they are capable only of answering exam style questions. By a mere twist of the question, making it open-ended, children are caught off-guard and are unable to answer the question, as there is "too little information given"

I enjoyed presenting this activity to the pupils as much as the pupils enjoyed it. As one child exclaimed during the discussion I had with the pupils, "this made understanding of factors so easy". True indeed, this activity was suited as an activity to launch the concept of factors. One constraint is the time factor. In our current system, teachers often find themselves trying to cope or catch up with the syllabus and scheme of work. To keep pace, often teachers resort to a teacher-centred style of learning.

The use of open-ended problems should be used more frequently in the classroom as it encourages pupils to think out of the box, to be more curious and be aware of the possibilities of more than one solution."

Mrs. B (Rectangle Problem)

"It was an enriching experience for the first time and the pupils enjoyed it too. I thought to a certain extent this activity has given them a certain level of confidence and also helped them to explore instead of just solving routine problems.

There was reinforcement of the concept of triangle during class discussion as they discovered they have used the wrong dimensions. Some groups after getting one answer stopped working as they have been doing problems with only one answer. I guess the teaching of math is different nowadays and pupils as well as the teacher need to change their mindset in solving problems

I felt that these open-ended problems could be introduced to more teachers and even implemented in the Math textbooks. As much as Math is concerned teachers still need to teach the fundamental concepts and with these knowledge pupils should be encouraged to explore amazing discoveries. At the end of the day, I guess what I really want is to teach them how to fish and not just to provide them with the fish!"

Conclusion

Due to the limitation of space here, there are several other interesting teachers' questions and pupils' responses that could not be accommodated in this paper. From the samples that we have presented here, the teachers who have used such short open-ended questions in their classrooms found many advantages that enhance their professionalism. It is an approach that enable them to teach mathematics that aligns with the national vision of "Thinking schools, learning nation" where the focus is on thinking and reasoning that characterize the shift from practicing isolated skills towards developing rich network of conceptual understanding. From the evidence presented in the pupils' work, the pupils were capable of communication in mathematics using words, diagrams, pictures or manipulatives. Pupils in presenting their solutions to others could compare and examine each other's method and discoveries from thence they could modify and further develop their own ideas in creative ways.

Note: The author wishes to thank all the teachers who participated in the in-service programme and especially those teachers whose feedback and pupils' work are shown in this paper.

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Connecting Mathematics and Science

By Low-Ee Huei Wuan, SP

On 17 August 2004, a group of 30 Mathematics teachers and lecturers gathered at Singapore Polytechnic (SP) for a workshop jointly organised by AME and SP, on making connections between the major subjects in our curriculum, Mathematics and Science.

The speaker, Mr Russell Brown, is an Australian Mathematics and Science teacher from Bendigo Senior Secondary College, Victoria. He shared with the participants how they could engage their students by making learning relevant, practical and fun in the classroom.

The activities as Mr Brown had shown, required only minimal preparation and expertise on the part of the teacher. Resources used were not sophisticated but basic household items such as like nail varnish remover, vinegar, baking powder and methylated spirit.

Mr Brown also demonstrated to the audience, the sequence from the collection of data with a datalogger to the plotting of graphs with a graphing calculator. In this way, teachers are able to relate real life data to graphs, and subsequently lead students to form appropriate mathematical models such as differential equations.

Judging from the fun the participants had, matching their movement to a drawn graph and vice versa, they have certainly taken away from the workshop more than just mere connections.

Square Numbers: Their Bonds with Odd Numbers

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Introduction

Students often recognise 1, 4, 9, 16, 25,... as the sequence of square numbers because the first term 1 is 1^2, the second term 4 is 2^2, the third term 9 is 3^2 and so on. However, not many may be familiar with another description of these square numbers in terms of a series of consecutive odd numbers. This article describes an activity which secondary school students may use to help them explore and establish this relationship. Students manipulate with abstract objects of 1^2, 2^2, 3^2, 4^2, 5^2,... to get the corresponding sequence of square numbers – 1, 4, 9, 16, 25,... In this activity however, students manipulate concrete objects – square tiles – and through manipulating such concrete objects, students get a sense of the pattern underpinning the structures they have constructed. Once students get a sense of the pattern underpinning these structures, they can begin to articulate this pattern verbally and then to generalise and record this pattern symbolically. This activity offers students the opportunity to link numerical representations of numbers with their corresponding geometric structures.

Teaching Points

According to Mason (1996), the primary aim of teaching mathematics is to expose students to a fundamental process of mathematical thinking - *generalising*. To achieve this aim, he proposes a helical model where students are given the opportunities to encounter and explore an idea. Through manipulation of the objects embedded in the idea, students begin to get a sense of the pattern underpinning this idea and are able to generalise the pattern. The phase of encountering an idea and exploring it is known as *manipulation*. The next phase of getting a sense of the pattern occurs when a pattern is observed from manipulating the idea. Lastly, the phase when the pattern is represented verbally, diagrammatically or symbolically is known as *articulation*.

Based on Mason's model, the activity described in this article is designed to allow students to manipulate objects to generate a sequence of growing squares, to describe how they form the growing squares and then to generalise the patterns they see. The aims are manifold:

To get students to
(a) focus on the manipulation of objects,
(b) use language to describe what their procedures are, and
(c) record the rules symbolically.

Also by doing so, students will link number patterns to their geometric form. From the algebraic aspect, this activity will additionally help to deepen students' understanding of the use of letters to represent numbers as well as structural equivalence. By structural equivalence, it is hoped that, for instance, 3^2 will be perceived as $3^2 = 1+3+5$ rather than just $3^2=9$.

The Activity

With teachers' guidance, students will work through a series of tasks to answer the question – *Find the sum of the first n consecutive odd numbers.*

Given the following diagrams of three squares, students will perform the tasks given in the table below.

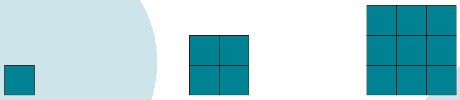
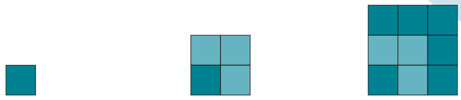


Figure 1 Figure 2 Figure 3

Manipulating and getting a sense of the pattern	1. Use square tiles to make these figures. 2. Describe how you would make Figure 2 from Figure 1, Figure 3 from Figure 2, Figure 4 from Figure 3.
Articulating the pattern	3. Without using the square tiles, describe how you would make Figure 10.
Generalising the pattern	4. Find the sum of the first n consecutive odd numbers.

If different coloured square tiles were to be used, students may be able to see another pattern and to generate relationships that were not obvious. For example, the following figures could be formed using tiles of two different colours.



One way to describe the growing pattern is

Number of tiles in Figure 1 = 1^2 (equation 1)
Number of tiles in Figure 2 = 2^2 = 1+3 (equation 2)
Number of tiles in Figure 3 = 3^2 = 1+3+5 (equation 3)

It is important for students to describe what they see in Figure 3 as a series of growing consecutive odd numbers (e.g., 1+3+5) and not as an outcome (e.g., 9 or 4+5). This mode of description should continue for Figures 4, 10 and so on. Additionally, teachers can encourage students to describe the structural equivalence of equation 3. From Figure 3, students may be able to recognise that the sum of the first three consecutive odd numbers is 3^2 – *the square of the figure number.* To consolidate the pattern they see, teachers can encourage students to test the pattern with Figures 4 and 10. Further consolidation may take place if students are asked to identify the figure number given the series of consecutive odd numbers such as 1+3+5+...+23. In this case, Figure 12 is the answer. With this, a link could be made between the figure number and the number of consecutive odd numbers in the series as well as between the first and last terms of the series. For example, the number of consecutive odd numbers in 1+3+5+...+23 is 12 which is the figure number and this figure number is also equivalent to the mean of the first and last terms of the series. Hence, given another figure number, the last term of the series of this figure could be determined. For example, given Figure 15, the last term of the series should be 2X15-1=29 and ideally this should be provided by students without having to list the entire series for Figure 15.

So, what is the sum of the first n consecutive odd numbers? The answer is clearly n^2! However, to find the sum of 1+3+5+...+357 is not so trivial. In order to answer this question, it is necessary to find out the number of terms in the series. This is where it is important to identify that the last term of the first n consecutive odd numbers is $2n$-1. In this case, $2n$-1 is equivalent to 357. With this in place, the value of n can be found and hence, the sum of 1+3+5+...+357 can also be determined.

Conclusion

Rather than describing 1, 4, 9, 16, 25,... simply as a sequence of square numbers, this article provides another interpretation of these square numbers from both the geometric and algebraic aspects with an exploratory activity. By using tiles to form squares, it is possible to show geometrically that every square number can be expressed as the sum of consecutive odd numbers. So, the term n^2 can be expressed algebraically as the sum of the first n consecutive odd numbers (i.e., $n^2=1+3+5+\ldots+(2n-1)$). Apart from this relationship and the fact that the last term is $2n$-1 , students may observe that n^2 can also be expressed as the square of the mean of the first and last terms of the sequence to be summed (i.e., $n^2=\left(\frac{\text{first term + last term}}{2}\right)^2$). As these patterns can be quite abstract to learn, it is therefore hoped that through manipulating and exploring objects as well as getting a sense of the patterns from it, students are able to connect better both the geometrical and algebraic representations of square numbers.

Reference

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On Teaching Vectors and Relative Velocity

Toh Tin-Lam, with contributions from participants of the in-service workshop on **V**ectors and **R**elative **V**elocity
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Relative velocity was introduced in the pure mathematics component of the GCE "O" Level Additional Mathematics syllabus in 2001. Previously, this topic was subsumed under the particle mechanics section in the earlier Additional Mathematics syllabus. Questions then tended to focus on the mechanical drawing of vector diagrams and the extensive use of geometry and trigonometry. On close examination of the recent GCE "O" Level syllabus, one would find the focus to be rather different now.

With this backdrop, I would like to discuss two aspects in the teaching of relative velocity : the intuitive understanding of the concept of relative velocity and the use of '**i-j**' vector notation in this topic.

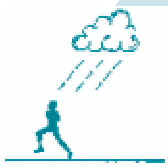
On this note, I shall begin by illustrating some numerical problems contributed by the in-service participants.

Intuitive Understanding of the Concept of Relative Velocity

Before introducing $\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B$, the formal definition of the velocity of A relative to B, it would be helpful to allow students to appreciate that $\mathbf{V}_{AB}$ is in fact how the observer B feels when A is moving (via a software such as a GSP applet, for example, see [**3**]). Since B is unaware that he is also moving, so what he perceived will be different from the observation made by any stationary observer A. Of course, if B is at rest (relative to the Earth), it would reduce to a trivial case of $\mathbf{V}_{AB}$ being the actual velocity of A, i.e. $\mathbf{V}_A$. Otherwise, the two quantities are not the same.

Consider the following question (the original source of this question is a particular Physics textbook, which the course participant had forgotten).

Example 1.
Rain is coming at an angle of 30° to the vertical plane at a speed of 4 m/s. A man is walking on horizontal ground as shown. How fast must the man walk / run, in the left direction, so that the rain will just miss hitting his back?



Let M denote the man and R, the rain. In the case of Example 1, the student solving this problem would realize that in order for the rain to *just miss hitting his back*, the rain must *appear to him* as if it was falling vertically, that is, $\mathbf{V}_{RM}$ must be acting vertically downwards. Following that, using either geometric construction or elementary trigonometric computation, the man's speed can be found easily. The reader can effortlessly verify that he must run at a speed of 2m/s[1].

Consider another question, which is quoted from a past year Cambridge Examination paper.

Example 2. (June 99)
The runner P starts her last 400 m lap of a race 14 m behind a runner Q. Each runner maintains a constant speed during the last lap and P wins the race with a lead of 1 m over Q. Given that the speed of P relative to Q during the last lap is 0.25 ms^{-1}, calculate the speed of P.

The runner Q in Example 2 perceived that P was 14 m behind him and finally completed the race 1 m ahead of him. So the total distance that Q perceived P had covered was 15 m. Since the speed of P relative to Q is 0.25 ms^{-1}, the time taken for P to overtake Q and win with a lead of 1 m = $\frac{15}{0.25}$ = 60 seconds. However, the actual time taken and the "relative" time taken are the same! Coming back to reality, P has traveled 400 m (within the time of 60 seconds). Therefore, the speed of P can be calculated as $\frac{400}{60} = 6\frac{2}{3}$ms^{-1}.

It is important to help students develop an intuitive feeling of relative velocity. To begin, it may be useful to start with questions such as in Example 1 and 2 before plunging into problems that require more sophisticated constructions and geometry.

The Use of 'i-j' Vector Notation in Relative Velocity

Some of the participants are worried about the abundant use of '**i-j**' notation to describe the position (displacement) and velocity of particles in vectors in the GCE 'O' Level examination questions. In such questions, students must be able to distinguish between displacement and velocity vectors. Students need to know the meaning of relative velocity in which the description is given in '**i-j**' notation. An example is appended below:

Example 3.
A ship sails from point A to point B, the position of B relative to A is (2400**i**– 600**j**) km , where **i** is a distance of 1 km due east and j denotes a distance of 1 km due north . Due to a steady current , the journey takes 6 hours . The velocity of the ship in still water is (120**i**+80**j**) km/h . Find the speed and direction of the current.

(Answer: 284.7 m/s; 085.9°)

In Example 3, students need to : (1) interpret the displacement and velocities in '**i-j**' notations; (2) understand the meaning of the terms *velocity in still water*, and other related quantities like the *true velocity of the ship* etc., and (3) apply the formula

Change in Displacement = (Time Taken) x Velocity,

which is a vector counterpart of the equation : *Distance traveled = Time Taken* x *Speed*.

Another nature of the problem includes determining the likelihood of two particles moving with constant velocity intercepting. If they meet, we would be requested to find the time of interception or otherwise, determine the shortest distance between them. A few questions contributed by the course participants are appended below.

Example 4.
Let **i** and **j** denote a distance of 1m along the East and the North respectively. A thief was running off at 2**i** m s^{-1} when he heard a growl from a menacing dog 5 m away, at a bearing of 150°. Wherever a vector is required as an answer, write it in the form of (a**i** + b**j**).

(i) Write out the position vector of the dog, relative to the thief's position.
(ii) Write out the displacement vector of the thief at the end of the 3rd second after he heard the dog.
(iii) Assuming the dog managed to bite the thief's leg 3 seconds after the thief heard it growl, find its displacement vector for these 3 seconds.
(iv) Calculate its speed, given that it ran at a constant speed in a straight line.

(Answers: 1(i) $\frac{5}{2}(i-\sqrt{3}j)$ (ii) 6im (iii) $\frac{1}{2}(7i-5\sqrt{3}j)$m (iv) 1.86 ms^{-1})

(Footnotes)
[1] Based on the fact that it is a Physics question, I believe the intention of the author is to test the students' ability in understanding the concept of resolution of vectors. However, for the teaching of relative velocity, our objective would be to instill in our students a more intuitive concept of relative velocity.
[2] While it is stated in the syllabus that problems of closest approach will not be tested, the entire idea illustrated in Example 7 is by the algebraic approach rather than by the relative velocity method. Hence the question can be linked to calculus or quadratic expressions.

Example 5.
A ship A is traveling with velocity $20j$ km/h. At 12 00 hour, it is at the point with position vector $\frac{50}{\sqrt{2}}i - \frac{50}{\sqrt{2}}j$ km. A second ship B is at the origin. If ship B steers at 25 km/h so as to just intercept ship A, find
(a) the direction in which B must travel
(b) the time when the interception takes place.

(Answer: 100.9^0; 13 26h)

In solving problems like Example 4 and 5, students need to be familiar with the concepts and apply the formula : at any time *t* after the movement of an object with constant velocity **v**, its position **r**(*t*) can be expressed as

$$\boxed{\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{v}} \qquad \text{(1)}$$

where **a** is the position vector of the starting point of the particle. The rest of the problems can be solved algebraically – even without knowing any concept of relative velocity!

However, the tediousness (in terms of algebraic computation) involved in solving the questions such as in Examples 4 and 5 above, can be reduced by using relative velocity :

> *Two particles* A *and* B *moving with constant velocities and with starting points* A_0 *and* B_0 *respectively will eventually intercept if and only if* $\mathbf{V_{AB}}$ *is along the direction of the vector* $\mathbf{A_0B_0}$. *The time at which the interception takes place is given by* $\frac{|\mathbf{V_{AB}}|}{|\mathbf{A_0B_0}|}$.

The above method can be derived algebraically, see [2; Pg 63-64]. The derivation only involves some fundamental knowledge of vectors.

If the entire question on the motion of two particles is given in **'i-j'** notation as in Examples 4 and 5, it can be worked out mentally whether two particles moving with constant velocity will eventually intercept. We will leave it to the readers to try the above two examples.

Appended below are two more similar examples.

Example 6.
With respect to an origin, a car is at the point with position vector $11\mathbf{i} + (10\sqrt{2} - 10)\mathbf{j}$ and a van is at the point $(11\sqrt{2}-5)\mathbf{i} + k(4\sqrt{2}-3)\mathbf{j}$. If the car is traveling at a constant velocity $\sqrt{2}\,\mathbf{i} + \mathbf{j}$ and the van is traveling at a constant velocity $2\mathbf{i} + \sqrt{2}\mathbf{j}$, find the value of *k* such that the two vehicles meet, giving your answer in the form $\frac{a+b\sqrt{2}}{c}$ where *a*, *b* and *c* are integers.

(Answer: $\frac{19+10\sqrt{2}}{23}$)

Example 6 above requires students to deal with surdic expressions in a problem involving kinematics / relative velocity. Example 7 requires students to work with the situation in which the two particles do not intercept. Hence the time of closest approach and the shortest distance between the two ships are required.[2]

Example 7.
The *RSS Courageous*, a navy ship is at the point with position vector 20**j** km and an oil tanker *Texan One* is at the point with position vector (-100**i** + 20**j**)km, where **i** and **j** denote a distance of 1 km along the east and north respectively. *RSS* Courageous is traveling at a velocity of 30**j** km/h and *Texan One* is traveling at a velocity of (10**i** – 20**j**) km/h. Will the two ships collide? If yes, find the time of collision; otherwise, find the time at which they are nearest together. Find this closest distance between the two ships.

(Answer: the ships are closest when *t* = 1 h and the shortest distance between them is $\sqrt{9000}$ km)

To solve Example 7, one way would be to prove that the two ships do not intercept (by either the algebraic means or the Relative Velocity approach) and then proceed by finding an expression for the distance between the two ships for time, *t*. To find the shortest distance between the two ships, one needs to use either calculus to minimize the expression of *t* or by the algebraic approach of completing the squares (since the expression in terms of *t* is a quadratic expression).

Conclusion

We have discussed the two aspects in the teaching of relative velocity : the intuitive understanding of the concept of relative velocity and the use of the **'i-j'** vector notation in this topic. We have also included some questions constructed by in-service course participants. For teachers who would like to acquire more resources on vectors and relative velocity, the references indicated below may be of help.

References

[1] National Institute of Education, *Teaching of Vectors, Kinematics and Relative Motion*, published as Chapter 14 in *The Green Book: Resources and Ideas for Teaching Secondary Mathematics*, National Institute of Education, 2004.
[2] Toh T.L., *Vectors and Relative Velocity*, McGraw-Hill, 2004.
[3] Toh T.L., *On using Geometer's Sketchpad to teach relative velocity*, Asia-Pacific Forum on Science Learning and Teaching, Vol 4(2), Article 8, 2003.
In http://www.ied.edu.hk/apfslt/v4_issue2/toh/

FORTHCOMING EVENTS FOR YOUR ATTENTION!

	Talk / Workshop	Speaker	Audience	Date & Time	Venue
1	Extendig Primary–Level Mathematics Activites to Promote Problem Solving	A/P Douglas Edge	Primary school teachers	30 October 2004	Maha Bodhi School
2	AME Session at ATCM: IT in Action				
	• Use of IT in Maths Revision (Maths Autograding Program) by River Valley High • Enriching the Classroom Experience Digitally by Greenview Secondary School • The Dancing Multiplier by Hougang Primary School	*school will showcase their IT initiatives	Participant of ATCM 2004	Tentatively scheduled on 15 Dec 2004	NIE

*AME is still sourcing for more schools to showcase how they have successfully incorporated IT in the teaching and learning of Mathematics. If you are interested in participating in the above event, please send an abstract of no more than 200 words, via email to Huei Wuan at this address, lowhw@sp.edu.sg

Contribution Invited

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