ASSOCIATION OF MATHEMATICS EDUCATORS

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Maths Buzz

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Top News of this Issue

Mathematics Teachers' Conference

Theme: Assessment

2nd June 2005
National Institute of Education

Dear Mathematics Teachers,

The theme of this conference is in line with the present emphasis on alternative assessment in our classrooms. There are different techniques used in alternative assessment to serve different pedagogical objectives, for example, performance assessment to assess how well pupils are able to apply mathematics to solve real problems, journal writing to encourage them to communicate mathematical ideas to others, and portfolio to allow pupils to showcase evidence of their mathematical understanding in different contexts. These uses require special knowledge and skills. The lectures and workshops that AME and MME have planned for this conference will expose you to the state-of-the-art techniques of alternative assessment and the research done will shed insights into how they work out in classrooms in Singapore as well as in other countries. The conference will also enable you to establish a network with fellow mathematics teachers who are experimenting with similar assessment techniques, and we encourage you to strengthen your collegial network after the conference as mutual support is an important success factor for your professional development.

We look forward to seeing you at the conference and trust that your active participation will make it a successful and memorable one.

A/P Berinderjeet Kaur
President,
Association of Mathematics Educators

A/P Wong Khoon Yoong
Head, Mathematics &
Mathematics Education
National Institute of Education
Nanyang Technological University

Highlight of the Programme includes:

Keynote I: Assessment – Policy, Practice, Practicalities and Praxis
by Prof. David Clarke (Uni. of Melbourne)

Keynote II: Assessment – Some Insights from Classrooms in Singapore
by Dr Fan Lianghuo (NIE)

Concurrent Workshops

General

A Structure for Quality Mathematics Instruction and Assessment
(Prof David Clarke, Uni. of Melbourne)

Scoring for Reliability and Consistency in Assessment
(Ms Lee Yim Ping, MOE)

Primary

Using Short Open-ended Questions to Develop and Assess Mathematical Thinking
(A/P Foong Pui Yee, NIE)

Conducting Investigative Tasks or Mini-projects in the Primary Classrooms
(A/P Koay Phong Lee, NIE; Leong Kok Fen & Eliza Lim Pei Lin, Fuhua Pri Sch)

Journal Writing in the Primary Classroom
(Juliana Ng, MOE)

Secondary

Journal Writing in the Secondary Classroom
(A/P Berinderjeet Kaur, NIE & Mdm Fauziah Ahmad, Westwood Sec Sch)

Conducting Self-assessment Tasks in the Secondary Classrooms
(Mdm Teo Soh Wah, NIE)

Conducting Investigative Tasks or Mini-projects in the Secondary Classrooms
(Ms Ng Luan Eng, NIE; Miss Linda Teo Lian Eng & Miss Leow Hwee Fen, Anderson Sec Sch)

Writing Good Assessment Items in Mathematics at the Secondary Level
(Ast/P Dindyal Jaguthsing, NIE)

Junior College

Challenging Mathematical Problems
(Ast/P Toh Tin Lam, Ast/P Dong Fengming & Ast/P Lee Tuo Yeong, NIE)

Mathematical Investigations as an Alternative Assessment at Junior College Level
(Ast/P Ng Wee Leng, NIE)

Panel Discussion: Chair: A/P Berinderjeet Kaur (President, AME)

Panelists: Prof. David Clarke *(Univ. of Melbourne)*,
Dr Fan Lianghuo *(NIE)*,
A/P Wong Khoon Yoong *(Head MME, NIE/NTU)*
A/P Lee Peng Yee *(NIE)*

Mathematics in Paper Folding and the Haga Theorem

Yoong Liang Teen, Catholic Junior College

Introduction

In paper folding, it is easy to obtain sides of length 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ..., $\frac{1}{2^m}$, where m is an integer. However for fractions like $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, ... where denominators are not in the form of 2^m, it may appear impossible. Kasahara and Takahama (1999) in the book, *Origami for the connoisseur* demonstrated a method for obtaining the length $\frac{1}{3}$ of a unit using the concept of similar triangles

Procedure to fold a length, $\frac{1}{3}$ of a unit

Assuming that the length of a square piece of paper is 1 unit, the paper is first folded into half to obtain the halfway mark A. It is then folded with a corner touching the side BG, as indicated in the diagram. In the process three similar triangles △ABC, △FED and △FGA are created.

Note that ∠ACB = ∠EDF = ∠FAG and
∠BAC = ∠EFD = ∠GFA.

Since A is the mid-point of GB, AB = $\frac{1}{2}$ unit.

BC + CA = 1 unit. Let BC = x unit. Then CA = 1 – x unit.

Applying Pythagoras Theorem on △ABC,

$AB^2 + BC^2 = AC^2$, and $x = \frac{3}{8}$

Hence AB = $\frac{1}{2}$ = $\frac{4}{8}$, BC = $\frac{3}{8}$ and AC = $\frac{5}{8}$.

Therefore AB: BC: AC = 4: 3: 5.

Since △ABC is similar to △FGA,

so FG : GA : FA = 4 : 3 : 5.

We know that GA = AB = $\frac{1}{2}$ = $\frac{3}{6}$,

then FG = $\frac{4}{6}$ = $\frac{2}{3}$ and FA = $\frac{5}{6}$.

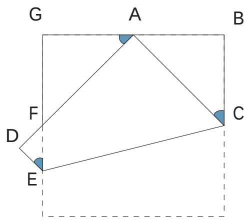


Diagram is not drawn to scale.

If FG = $\frac{2}{3}$, then the remaining length of that side of the paper would give $\frac{1}{3}$. This method is called Haga Theorem.

Procedure to fold the lengths, $\frac{2}{5}$ of a unit

The Kasahara and Takahama (1999), extended Haga's Theorem to fold lengths of $\frac{2}{5}$ of a unit. To obtain the latter, instead of having the corner A at the mid-point of GB, the corner is folded such that AB is $\frac{1}{4}$ of the length of GB. Using the same idea as above, we obtain FG = $\frac{2}{5}$

Since △ABC is similar to △FGA, so to obtain the length of $\frac{1}{5}$, we fold this length FG into half.

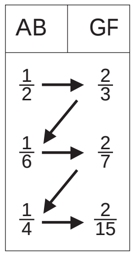
Procedure to fold the length $\frac{1}{n}$ for an integer, *n*

Using the idea of similar triangles, lengths of any fractions can be systematically generated by performing a series of calculations and folding, assuming that AB takes the length $\frac{1}{n}$ where n = 1,2,3,....

Listed in the table at the side are the values of GF given AB = $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{15}$

AB	GF
$\frac{1}{2}$	$\frac{2}{3}$
$\frac{1}{3}$	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{2}{5}$
$\frac{1}{5}$	$\frac{1}{3}$
$\frac{1}{6}$	$\frac{2}{7}$
$\frac{1}{7}$	$\frac{1}{4}$
$\frac{1}{8}$	$\frac{2}{9}$
$\frac{1}{9}$	$\frac{1}{5}$
$\frac{1}{10}$	$\frac{2}{11}$
$\frac{1}{11}$	$\frac{1}{6}$
$\frac{1}{12}$	$\frac{2}{13}$
$\frac{1}{13}$	$\frac{1}{7}$
$\frac{1}{14}$	$\frac{2}{15}$
$\frac{1}{15}$	$\frac{1}{8}$

As observed from the table, if AB = $\frac{1}{2p}$, then GF = $\frac{2}{p+1}$, and if AB = $\frac{1}{2p+1}$, then GF = $\frac{1}{p+1}$



Using this table, we can work out a systematic way to fold lengths of the form $\frac{1}{n}$. For example, if we want to fold a length of $\frac{1}{15}$. We first start with AB = $\frac{1}{2}$ which give us GF = $\frac{2}{3}$. Then follow the steps as illustrated int he chart finally we obtain GF = $\frac{2}{5}$ and we fold this GF into as half. So theoretically, any length of the form $\frac{1}{n}$ where n is an integer can be folded using the above procedure.

Conclusion

On the whole, it is interesting to find that there are ways to divide a length of a side of a paper in other fractions beside the usual $\frac{1}{2^m}$ where m is an integer and the method is also not too difficult for smaller odd numbered denominators like $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{7}$.

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**This article was written under the supervision of Professor Lee Peng Yee.*

An Overview of Proportional Reasoning

Jaguthsing Dindyal, National Institute of Education

Very often we come across the terms *proportion* and *proportional reasoning* in mathematics education. These terms may not be quite clear for everybody. In what follows, an attempt is made to clarify the terms using some examples.

What are proportions? Proportions are statements that two ratios are equal, and understanding the underlying relationships in a proportional situation and working with these relationships has come to be called as proportional reasoning (National Research Council [NRC], 2001). Proportional reasoning is generally regarded as one of the components of formal thought. In Piaget's theory, "proportional reasoning was the hallmark of the formal operations stage of development" (Lamon, 1993, p. 41). It has been described as the capstone of elementary school arithmetic and the gateway to higher mathematics, including algebra, geometry, probability, statistics, and certain aspects of discrete mathematics (NRC, 2001).

Post, Behr, and Lesh (1988) claimed that traditionally, proportional situations have been embedded in missing-value problems, a/b = c/x, where x is to be found. It should be noted that research on proportional reasoning at the middle school level has mostly been based on direct proportions and not inverse proportions. Proportional reasoning has also been used to compare two given rate pairs in which the task is to deduce which rate pair is greater. Problems on proportional reasoning can be solved by using either the *unit rate* method or the *factor of change* method. The unit rate method consists of scaling both ratios to obtain equivalent ratios in terms of a single unit, whereas the factor of change method, consists of finding a common multiple of one of the units for both of the ratios involved in the problem. For example, if three pencils cost \$5 then the unit rate is \$$\frac{5}{3}$ per pencil but 18 pencils will cost 6×5 = \$30 by the factor of change method, where 6 = 18÷3 or 3: 5 = 18: 30.

Students usually meet with three types of problems on proportional reasoning: (1) missing value problems, where three pieces of information are given and the task is to find the fourth missing piece of information; (2) numerical comparison problems, where two complete rates/ratios are to be compared; and (3) qualitative prediction and comparison problems which require comparisons on specific numerical values (Ben-Chaim, Fay, Fitzgerald, Benedetto, & Miller, 1998; NRC, 2001). One must guard against the fallacy that proportional reasoning involves only quantitative reasoning. Cramer, Post, and Currier (1993) claimed that we cannot define a proportional *reasoner* simply as one who knows how to set up and solve a proportion, instead they strongly pointed out that a focus on both the quantitative and qualitative aspects of proportional reasoning is essential in schools. Others like Post, Behr, and Lesh (1988) have also argued that proportional reasoners should be able to distinguish between proportional and nonproportional situations. These authors also added that proportional reasoning involves a firm grasp of various rational number concepts such as order and equivalence, the relationship between the unit and its parts, the meaning and interpretation of ratio, and issues dealing with division. Unless students have this facility with rational numbers they will be at risk of doing their operations wrongly. Quite often textbook problems tend to use the so-called 'nice numbers' and this leads students to look always for the 'nice numbers' answers as well. A proportional reasoner should not be influenced by context or numerical complexity. Students should be cautioned about such misguided reasoning to look only for the 'nice numbers'.

The development of proportional reasoning should be extended over several middle grade years. This point was also advocated by the NRC (2001) report, that proportional reasoning is not acquired all at once; rather it is through a relatively long and lengthy process that middle grade students come to understand proportionality and use it appropriately in problem situations. But proportional reasoning is not only a problem that middle grade students face. Some students still have problems with proportional reasoning at college level. Research by Lawton (1993) showed that college students had many difficulties solving problems on proportionality.

Below, I detail three instructional tasks which can be used to develop proportional reasoning. These tasks may be used for Lower Secondary. The three tasks emphasize the various types of problems that students will usually meet at this level.

Task 1

Mini Chips cookies cost$1.39 per package and has 17 cookies with a package weight of 400 g. Duffy's Delights cost $2.29 per package of 10 cookies with a package weight of 700 g.

a. Decide which unit you would choose to compare the different brands of cookies. Explain why you choose that unit.
b. Find the price per unit of each brand of cookie.
c. Compare the unit price of the two brands of cookies. Is one brand the better buy? If so, explain why. If not explain why the brands are equally good buys.

Source: MathScape (1998)

Task 2

Arvind and Mariah tested four juice mixes.
Mix A has 2 cups of concentrate and 3 cups of water.
Mix B has 1 cup of concentrate and 4 cups of water.
Mix C has 4 cups of concentrate and 8 cups of water.
Mix D has 3 cups of concentrate and 5 cups of water.

a. Which recipe will make juice that is the most "orangey"? Explain your answer.
b. Which recipe will make juice that is the least "orangey"? Explain your answer.
c. Assume that each camper will get half a cup of juice. For each recipe, how much concentrate and how much water are needed to make juice for 240 campers? Explain your answer.

Source: Connected Mathematics Project (1997.)

Task 3

On a map a distance of 2 centimeters represents an actual distance of 5 kilometers. Draw a line segment, which will represent an actual distance of (i) 15 kilometers, and (ii) 22.5 kilometers on the map.
Explain why the line segment that you have drawn represents the actual distance.
Fill in the table given below

Distance on map (cm)	2	3		9.5
Actual distance (km)	5		17.5	

a. Write a formula that can be used to determine the distance on the map (M) if the actual distance (A) is known.
b. Test your formula using the number pairs in the table. Use your formula to determine the actual distance on the map if the actual distance is 72 miles.
c. Graph the data from the table on a pair of coordinate axes and connect the data points.
d. Describe what the graph looks like.
e. What is the actual area in square kilometers represented by an area of 1 square centimeter on the map?

Comments on the tasks

It is assumed that the students who are going to do these tasks have had ample experience with rational numbers and the ratio concept. It is also expected that the students have had some facility with graphical representations. All students are expected to use a calculator to work on the tasks. It will be unwise to say that using only the three mathematical tasks described above will give a sound understanding of proportional reasoning to the students. Researchers agree that proportional reasoning develops over a period of time, and so it is expected that tasks such as the ones described above would be used over a fairly long period of time. Research has identified that proportional reasoning involves both quantitative and qualitative modes of thought (see Cramer, Post, & Currier, 1993) and accordingly in each of the three mathematical tasks, students are asked to explain their thinking and not just give a numerical answer.

The three tasks described above help students to develop mathematical proficiency, as described by NRC (2001). The tasks cater for a wide range of abilities of the students. For example, in Task 1, weaker students may focus on the price or the number of cookies as the unit of comparison. However, this will provide an excellent opportunity for discussion in the classroom as to what constitutes a unit and why is one better than the other. In Task 2, different students can come up with different strategies for solving the problem. Some may use the *unit rate* strategy, while others may use the *factor of change* strategy. Thus the tasks give opportunities to most students to achieve a certain measure of success. The last part of Task 3 will be demanding for the average student, but this can be used to challenge the stronger students in class. Students are free to explore their own ideas about solving the problems. Task 3 allows students to make connections between the different representations when dealing with the ratio concept and proportional reasoning. There is a tabular representation, an algebraic representation, and also a graphical representation. The teacher may ask additional questions about the nature of the graphical representation in cases involving direct proportion. The tasks provide the teacher with valuable information about the students' thinking when dealing with proportions. Rather than students working individually, the teacher should allow them to work in small groups. The interaction within the groups can be a good source of discussion which may eliminate some of the misconceptions that students may have.

From a teaching point of view, one of the problems that the teacher may face is a lack of prerequisite knowledge for dealing with the tasks. Students may have a very poor grasp of the rational number concept and may have problems with the ratio concept as well. This can hinder the students' progress on these three tasks. Hence, before embarking on these tasks it is very important for the teacher to ascertain that these prerequisites are met. It is important that students keep track of what they are using as a unit to compare and give sufficient reasons for doing so. The teacher must ask questions to lead students in desired directions without actually giving away the solutions. In Task 3, the lack of experience of the students with a graphical representation can be a problem. However, this task can be used to emphasize some graphical competencies as well. Another problem which has been identified in the research literature is that of the transition from additive to multiplicative reasoning. Teachers must be aware that some students may be still much tied to an additive type of reasoning from their earlier work in arithmetic and consequently face difficulty with proportional reasoning which uses multiplicative reasoning.

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When you next look at flowers …

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This is the first of a series of articles featuring the famous Fibonacci numbers. The article begins with an introduction to Fibonacci numbers and the all important golden ratio ϕ. We then provide some examples of where the Fibonacci numbers can be found in the plant kingdom and this paper concludes with an explanation why it is more common for plants to bear flowers with five petals.

The Fibonacci sequence

Leonardo of Pisa better known as Fibonacci was an Italian mathematician of great stature. In his voluminous work *Liber Abacci* (1228) he presented a simple hypothetical problem which investigated the number of rabbits born from a single pair of rabbits should they reproduce at a certain rate. The problem goes like this: A pair of young rabbits was placed in an enclosed place. If this pair of rabbits take a month to mature before they can reproduce, how many pairs of rabbits will there be in a year if each new pair of rabbits will produce a new pair of rabbits in the second month. Table 1 shows how the hypothetical rabbit population grows.

Table 1 The number of rabbits at the end of one year

Month	Adults	Babies	Youngsters	Total
January	0	0	1	1
February	1	1	0	2
March	1	1	1	3
April	2	2	1	5
January 1	144	144	89	377

Fibonacci's investigation gave birth to a simple number sequence that now bears his name - the Fibonacci sequence:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

This sequence starts with 1, each subsequent term in the sequence is the sum of the preceding two terms.

There are other numerical sequences such as $u_1, u_2, u_3, \ldots, u_n$ (1) in which each term equals the sum of the two preceding terms, for $n > 2$, $u_n = u_{n-1} + u_{n-2}$ (2) Recurrent sequences are those where each term of the sequence is defined as a function of the preceding ones. Recurrent process refers to the mode by which the terms are defined in relation to each other and equation (2) refers to the recurrence equation. Thus the Fibonacci sequence is an example of such recurrent sequences and the numbers are known as Fibonacci numbers.

The Fibonacci numbers are well known as they continually crop up in the plant and animal kingdoms. So what is it that makes these numbers so relevant to life? The answer lies in the manner in which the numbers in the sequence are related to each other and this relationship is expressed as the golden ratio ϕ.

The golden ratio

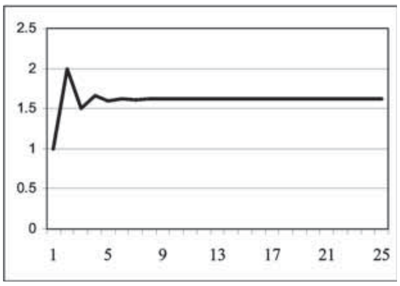

Figure 1: Ratio of the first 25 numbers

When the ratios of consecutive Fibonacci numbers are calculated, the ratios of the numbers stabilise around the value of about 1.618. Table 2 shows the ratios of the first 10 ratios while the graph in Figure 1 shows how the ratios for the first 25 numbers stabilise at about 1.618 which is known as the golden ratio ϕ.

Table 2 First ten ratios

Ratio Term number		1	2	3	4	5	6	7	8	9	10
Fibonacci numbers	1	2	3	5	8	13	21	34	55	89	144
$\frac{u_n}{u_{n+1}}$, for n>1		2	1.5	1.666	1.6	1.625	1.615	1.619	1.618	1.618	1.618

Where you can find Fibonacci numbers – the plant kingdom

The next time you are in a garden, spend some time and examine the flowers of plants. You may notice that while the number of petals on different types of flowers may vary; many flowers tend to have five petals (see Figure 2). However the *Cassine viburnifolia* have blooms with 5 and sometimes 4 petals on the same plant – a natural variation demonstrated by some plants. Most flowers have a Fibonacci number of petals (see Figure 4). Even if they don't, the number of petals is twice as large and from another sequence formed by applying the Fibonacci rule. Such sequences could be 2, 4, 6, 10, 16, … or 4, 7, 11, 18, 29, . Readers may want to refer to the book *1001 Garden Plants in Singapore* by Boo Chih Min, Kartini Omar-Hor and Ou-Yang Chow Lin for examples of plants bearing flowers with Fibonacci numbers of petals.


Cerbera manghas (Seashore Pong Pong) *Laurentia longiflora (Star of Bethlehem)*

Figure 2: Flowers with five petals

Figure 3: Cassine viburnifolia 5 petals (sometimes 4, the one on the right)

Acanthus sp. (Sea Holly) — *Tradescantia purpurea* (Purple Heart)

Figure 4: Flowers with Fibonacci number of petals

Cut an apple halfway across its equator. You will notice that the seeds of the apple are arranged as a five-pointed star (see Figure 5). The distance between the first and third points of the star is ϕ times the distance between the adjacent tips.

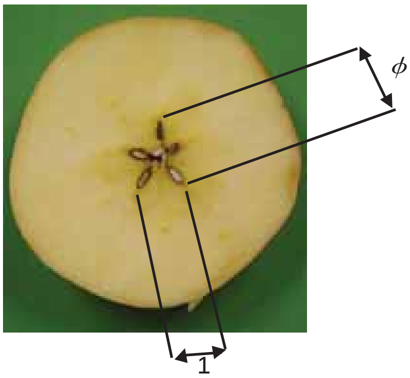

Figure 5: The golden ratio in apples

Investigate the arrangement of the prickly scales on pineapples and you will notice that the scales are arranged in spirals which run in a clockwise and anti-clockwise direction (see Figure 6). If you count them you will notice that there are 8 spirals going in one direction and 13 in the other.

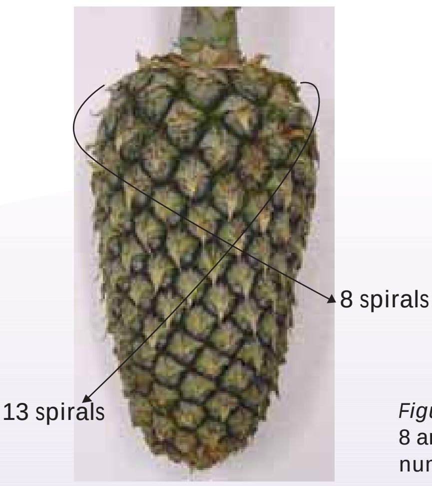

Figure 6: For pineapples, 8 and 13 are significant numbers

Is it by chance that the Fibonacci numbers are more commonly found in plants and is there a logical reason for their common occurrence?

Why flowers favour the Fibonacci numbers over others

The ancient Greeks were fascinated with the geometry of the regular pentagon and the inscribed five-pointed star. They showed that in regular pentagons with sides one unit long, the length of the sides of the five-pointed star formed by joining each vertex to the next but one is ϕ (see Figure 7).

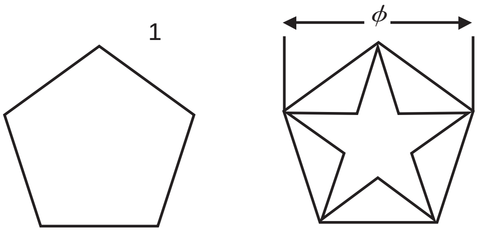

Figure 7: Regular pentagon of unit length and a five-pointed star of length

The flowers in Figure 2 all have five petals. Why is five so favoured by plants? Perhaps there is a link with the five petals found in flowers with the regular pentagon that fascinated the ancient Greeks. If you measure the angle between the first petal and the second, you will find the angle between them lies somewhere between 137^0 and 139^0. In fact the product of $1/\phi$ and 360^0 is 222.5^0 which is the complement of the angle between the petals, 137.5^0. The plant requires an arrangement such that the petals are best arranged to enhance the pollination process. So during flower development, the plant pushes the new petal as far away from the previous petal and yet still allows as many petals as possible to grow. By packing the petals at the golden angle of 137.5^0 allows the plants to do just that. After the emergence of the first petal, the second petal is at an angle of 137.5^0 from the first petal. The third petal grows at 137.5^0 from the second and so on till the fifth. The angle between the fourth and the first and the fifth from the second is 52.5^0. Figure 8 shows the configuration of a 5-petal flower which approximates very closely to the regular pentagon.

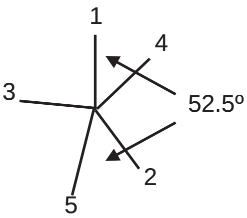

Figure 8: Configuration of a 5-petal flower

So the next time you look at flowers, check the number of petals. What Fibonacci number is exhibited by the flowers?

In the next Maths Buzz article we will look at the arrangement of leaves on stems of plants and its relation to the Fibonacci numbers and the golden ratio ϕ.

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M & M: An exploration of the links between Mathematics and Music

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Before an orchestral performance, the concert master will always lead the other musicians to tune their instruments prior to playing together. Now, some of you may begin to wonder what this example has got to do with mathematics. Indeed, very few people are aware that mathematics also lends itself to the performing arts, in particular, music. So, the aim of this article is to describe the connection between mathematics and music so as to broaden the teachers' horizons as well as to enrich their mathematical experiences. It is hoped that this article will serve as a springboard for teachers who like to explore this connection further with their students.

The Mathematics-Music Connection

The present ***musical scale*** is a series of twelve notes (made up of seven white keys and five black keys), which is recursive on a keyboard instrument such as a piano. A segment of a piano keyboard, showing an octave, is presented in Figure 1.

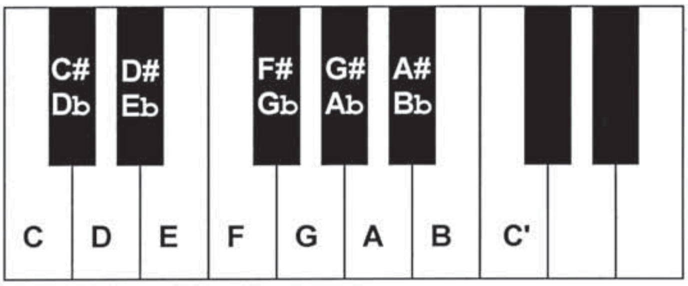

Figure 1. An octave on a piano keyboard

The white keys are given letter names from A to G in a cyclic fashion (that is, A B C D E F G A B C ...) while the naming convention of the black keys derives from the surrounding white keys (Taylor, 1989). For instance, the black key between C and D is named either C# or Db, depending on the harmonic context from which it is viewed. When the black key is viewed as a rise in pitch from C, then it is called C#. In a similar manner, it is named Db when viewed as a lowering of pitch from D. In general, the notation '#' denotes a rise in pitch whereas 'b' signifies a lowering of pitch.

Before the present musical scale was developed, there was what is typically known as the "Just Scale". The development of the "Just Scale" can be traced to ancient Greek contributions to mathematics and science (Calinger, 1999). Not many people are aware that it was actually the Greek mathematician, Pythagoras, whose name will evoke a recollection of his famous theorem relating the three sides of a right-angled triangle, who made a significant impact on the western music theory. He discovered that the notes which were most pleasant to the ears were produced by taking fractions of the frequency of the reference note.

Pythagoras found that when two stretched strings, one being half the length of the other, were plucked, the notes produced sounded similar except that the one generated by the shorter string was higher in pitch. The two notes are said to be an *octave* apart, which is an interval between twelve consecutive notes such as C and the next C. In terms of frequencies, the higher note produced by the shorter string has a frequency that is twice that of the lower note produced by the longer string. Pythagoras also discovered that when the shorter string was two-thirds, instead of one-half, the length of the longer string, then the note produced is said to be a *fifth*[2] higher than the note generated by the longer string. In other words, if the note produced by the longer string is C, then this shorter string will produce the G note, and its frequency is $\frac{3}{2}$ that of C. Similarly, when the length of the shorter string was three-quarters the length of the longer string which produced the C note, then the F note was heard, and its frequency is $\frac{4}{3}$ that of C. In music terminology, the F note is said to be a *fourth*[3] higher than the C note. In summary, Pythagoras discovered that the ratios of $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$ for the lengths of the plucked strings under the same tension gave rise to the octave, the fifth and the fourth respectively (Calinger, 1999).

Suggested Student Activity

Once the frequency ratios for these three intervals are known, teachers can lead students to compute the frequency ratios for other intervals such as from C to D, C to E, C to A and C to B as shown in the following table.

C to:	D	E	F	G	A	B	C'
Frequency ratio			$\frac{3}{4}$	$\frac{3}{2}$			$\frac{2}{1}$

In this article, two examples are provided to illustrate how the frequency ratios can be computed so that teachers can model the method to students to enable them to find the others. The first example involves determining the frequency ratio for C to the next C using the known facts that the interval from C to G has a frequency ratio of $\frac{3}{2}$and the interval from C to F has a ratio of $\frac{4}{3}$. Suppose the frequency of C is x, then the frequency of G above C is $\frac{3}{2}x$. Subsequently taking a fourth from G (that is, the next C above G) will show that the frequency of the next C is given by

$$Frequency = \left(\frac{3}{2}x\right) \times \frac{4}{3} = 2x$$

This calculation confirms that sequentially taking a fifth (that is, C to G) followed by a fourth (that is, G to next C) results in an octave, and it has demonstrated mathematically that the frequency of the next C is indeed twice that of C. This result is again consistent with Pythagoras' discovery.

The second example demonstrates the computation of the frequency ratio for C to D. What is interesting here is that the frequency ratio for C to D is similar to that for F to G, and to calculate the frequency ratio for the interval from F to G, we need to work from F to C, then to G using the transitivity property. The frequency ratio for C to F is $\frac{4}{3}$, and so the frequency ratio for F to C is $\frac{3}{4}$. Also given that the frequency ratio for C to G is $\frac{3}{2}$, the frequency ratio for F to G is therefore $\frac{3}{4}\times\frac{3}{2}=\frac{9}{8}$. Hence,

(Footnotes)

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2 A fifth refers to the musical interval, based on the Just Scale, between one note and another that is five notes away from it. For instance, E to B is a fifth because B is five notes away from E, and so is D to A.

3 A fourth refers to the musical interval, based on the Just Scale, between one note and another that is four notes away from it. For instance, D to G is a fourth, and so are E to A and F to B.

the frequency ratio for C to D is found!

Students can be subsequently asked to find the frequency ratios for the other intervals using the method as shown in the two examples. After they have found the required frequency ratios, they can be further challenged to find the frequency ratios of consecutive notes such as D to E, E to F, G to A, A to B and B to the next C. It will be helpful if teachers can remind students that the ratios for C to D, as well as for F to G, have already been determined. When these ratios have been computed, students can be asked to verify if the ratio of the frequencies of consecutive notes is a constant. To learn more about this and its implications in the different types of music, students can be encouraged to explore and search for more details through the internet or books. This is also where teachers can engage students in an interesting classroom discussion to talk about how the "Well-tempered Scale" was developed from the "Just Scale".

Final Curtain Call

In this article, we have only discussed an aspect of the mathematics-music connection: the temperament of the western musical scale. In a short article such as this, it is not possible to elaborate in great detail the mathematics behind the development of the Well-tempered Scale. The brief discussion in this article is just an attempt to introduce the mathematical aspect of music to teachers. To explore deeper into the interesting connection between mathematics and music, there are many worthwhile areas within the topic of music that deserve further investigation by both students and teachers. For instance, students can investigate why the interval between C and the next C is divided up into twelve equal parts, with each equivalent to the twelfth root of 2 instead of twelfths.

The recent implementation of educational initiatives such as Interdisciplinary Project Work advocates departing from compartmentalising different disciplines and moving towards an emphasis on the inter-connectedness of separate disciplines. Therefore, classroom activities such as the one described in this article can offer students a glimpse of the application of mathematics in other disciplines. Students who go through such an interdisciplinary activity will not only gain a better understanding of the mathematical concepts involved, but also develop an appreciation for the beauty and power of mathematics. In addition, the activity affords them an opportunity to articulate mathematical ideas clearly, thereby enhancing their communication skills as well. All of these are desired outcomes that are emphasised in the pentagonal framework of our Mathematics curriculum.

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Football and Mathematics – Win, Lose or Draw?

Eric Chan, National Institute of Education

Football is a game that continues to capture many a hearts. The World Cup, the Champions League, the Asian Cup are but a few of the many major tournaments that continue to draw people from different lands together to witness these matches. The recent Singapore victory in the Tiger Cup appeared to have revived interest in the local football scene since the country's departure from the Malaysia Cup in the 1980s. However, what have been dominantly captivating football lovers is the English Premier League (EPL) each year. This year is no different. The local coffee-shops strategize in drawing the crowds by screening live matches while the Singapore Pool offers legalized betting at their multiple betting outlets. At the home front, cable television screens at least 5 live matches per week. Those who hunger for such news get their fair share also in the newspapers. Football lovers cannot help but see how the league table unfolds after each match.

The league table is an interesting table to scrutinize. Using the league table to discuss football with upper primary students is a good opportunity to introduce them to the mathematics therein.

Team	P (Played)	W(Win)	D(Draw)	L(Lose)	F(Goals for)	A(Goals against)	Pts(Total Points)
Chelsea	26	20	5	1	49	8	65
Man United	26	16	8	2	43	16	56
Arsenal	26	16	6	4	58	30	54
Everton	26	14	6	6	31	27	48
Liverpool	26	13	4	9	41	27	43
Middlesbrough	26	11	7	8	41	35	40
Bolton	26	11	6	9	35	32	39
Charlton	26	11	5	10	30	36	38
Tottenham	26	10	6	10	33	30	36
Man City	26	8	9	9	31	27	33
Aston Villa	26	8	8	10	29	33	32
Newcastle	26	7	10	9	37	43	31
Portsmouth	26	8	6	12	29	38	30
Birmingham	26	7	8	11	29	33	29
Fulham	26	8	5	13	33	44	29
Blackburn	26	5	10	11	21	36	25
Crystal Palace	26	5	7	14	29	40	22
Norwich	26	3	11	12	26	49	20
Southampton	26	3	10	13	28	43	19
West Bromwich	26	2	11	13	23	49	17

Table 1: EPL Standing Table as at 8 February 2005

In Table 1 above, the league status stands as such as at 8 February 2005. In all, there are 38 matches to be played by each team on a home-and-away basis. The first column shows the 20 teams, the second column shows the number of matches played (P), the third shows the number of matches won (W), the fourth the number of matches drawn (D), the fifth the number of matches lost (L), the sixth the number of goals scored for the team (F), the seventh the number of goals scored against the team (A), and the eighth the total number of points chalked up by the teams (Pts). Each win earns the team 3 points, while a draw 1 point. Losing a match, logically, does not earn the team any point.

There are at least 10 ways (through questioning) that I can use the league table to formulate questions to promote mathematical thinking. Questions can be crafted based on several categories that increase in their degree of complexity. For example, questions that require students to read data from the table, compute, and even construct scenarios under specified parameters.

S/no	Question	Mathematical thinking involved
1	Which 3 teams have the worst goal difference record?	West Bromwich, which is at the bottom of the table, has a goal difference of 26 (computed from 23 – 49 = - 26) going against them. Norwich has the next worst record of -23, followed by a tie between Southampton and Blackburn with a goal difference of -15.
2	Which team has a better goal scoring average than Chelsea?	Looking at the "F" column, Arsenal has scored more goals (58) than Chelsea (49). On average, Arsenal scores 2.2 goals per match (computed from 58 ÷ 26) compared to Chelsea's 1.9 goals per match (49 ÷ 26).
3	If in the next 6 matches, Man U wins 3 and l oses 3 while Arsenal draws all 6, will Arsenal be better off?	By wining 3 matches, Man U will earn 9 points, taking her total to 65, while Arsenal will add 6 points to hit 60. Arsenal is now worse off by 5 points compared to 2 points currently despite not losing any of the next 6 matches.
4	From which team will they have absolutely no chance of catching up with Chelsea even if Chelsea loses all the remaining matches?	Teams that have absolutely no chance if based purely on mathematics, will be from Blackburn down the table. This is on the assumption that even if Chelsea loses all the remaining 12 matches (with 65 points in all), and Blackburn wins all the remaining 12 matches to get to 61 points.
5	Defensively, Arsenal has let in 30 goals, which is a not a good record. Why is Arsenal still 3rd in the table?	Arsenal has won more matches than those teams below it which accounts for the points and her current position. Though it has let in many goals, it has also scored many. It is a case of scoring more than the opponents and thus winning the matches.
6	From the current table, what accounts for the 22-point gap between Liverpool and Chelsea?	Liverpool won 13 matches while Chelsea won 20 matches. The difference is 21 points ((20-13) X3). From column D, Chelsea had drawn 1 more match than Liverpool, which accounts for the additional point.
7	What scenarios are possible in order for Manchester United (Man U) to be on par or overtake Chelsea at the quickest?	Man U now trails Chelsea by 9 points. Scenario 1: Chelsea loses the next 3 matches (loses 9 points) while Man U wins next 3 matches to take them on par with Chelsea. Scenario 2: Chelsea loses twice and draws twice in the next 4 matches (earns 2 points to take them to 67 points), while Man U wins next 4 matches (to take them to 68 points)
8	By which match can Chelsea become champion if they keep winning before the league ends?	With a 9-point gap against Man U, Chelsea can afford to lose 3 matches. Hence, if Chelsea continues to win until match 35, they would be crowned champion even if Man U wins all their remaining matches (provided Man U's goal difference is not superior to Chelsea's).
9	Can a simple algebraic expression be formulated to depict the number of points chalked up by each team?	Pts = 3W + D, where Pts is the number of points accumulated, W is the number of matches won, and D is the number of matches drawn.
10	From the table, can some figures be established to show the consistency of each team?	Taking the number of wins and dividing by the number of games, a "winning index" can be established. For example, Chelsea has a winning index of 0.77 while Man U and Arsenal both have a winning index of 0.62. The bottom team has a winning index of 0.08.

The questions crafted are not exhaustive, but it can generate interest in the classroom to tie mathematics with a popular sport. Particularly for students who may not be good in mathematics but love football, using the league table to generate discussion involving mathematics provides an avenue to ease the math anxiety and work towards building confidence in mathematics.

FORTHCOMING EVENTS FOR YOUR ATTENTION!

	Talk / Workshop	Audience	Date & Time	Venue
1	Mathematics Teachers' Conference	Teachers & Mathematics Educators	2 June 2005	NIE

Due to overwhelming response, application for the conference has been closed

Contributions Invited

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