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## President's Message...



Dear AME Members,

The Association of Mathematics Educators held its inaugural Mathematics Teachers' Conference this year on 2nd June. An overwhelming response was received for the conference. However we were only able to register 524 mathematics teachers from primary, secondary and junior colleges for the conference due to our limited planning and resources. Feedback gathered from the participants placed on record that the conference was a great success and that mathematics teachers were in need for an annual conference. The great success of this conference may be

attributed to the relentless hours of hard work put in by the sixth executive committee of the Association, strong support of the Director of National Institute of Education, Professor Leo Tan, and the generosity of our sponsors. The Association hopes to make the conference an annual event. The second Mathematics Teachers' Conference will be held in June 2006.

In our classrooms, "Touching Hearts, Engaging Minds" and "Teach Less, Learn More" must direct our actions. Our classroom pedagogies and interactions with our pupils merit attention so that we better prepare our learners for life. The Association through its many activities hopes to provide mathematics teachers with opportunities to develop themselves. Change in pedagogy will be brought about through Action Research and active participation in learning journeys by teachers. Mathematics teachers must look beyond syllabuses and textbooks when planning their lessons. They have to be mindful of research that is available to guide them in teaching and experimentation in their classrooms. For example, teachers must redesign their lessons if they fail to achieve their lesson objectives and not resort to re-teaching the same lesson during "remedial" lessons. Teachers must also get used to the idea of working in small collaborative groups to prepare their curriculum, engage in peer observations and be critical peers. Finally, as we make changes to teach better rather than teach for mere tests and examinations, may I take the opportunity to introduce you to the Standards for Excellence in Teaching mathematics in Australian Schools. This document is available at: <http://www.aamt.edu.au>

*A/P Berinderjeet Kaur*

President

Association of Mathematics Educators

## Coming Attraction!

Look out for details on the Mathematics Teachers' Conference, 1st June 2006 organised by the Association of Mathematics Educators, AME.

Sign up early to avoid disappointment.

# The world's best tool to stimulate mathematical thinking



One of last century's great physicists, John 'black hole' Wheeler famously summed up one of science's golden rules: never do a calculation before you know the answer. Another towering scientific figure of the last century, Nobel prize winner Enrico Fermi, gained folklore fame for his 'Fermi puzzles', irreverent estimation problems he set the world's best physicists.

Both men were alluding to the crucial skill of Estimation. Their comments and actions highlight two things: first, estimation can and does pose a challenge to even the world's best minds; and second, estimation is essential for scientists and mathematicians: one need to have a good feeling first for what the final answer will be before embarking on any detailed work. As this essay will argue, what's true for scientists is true for all of us: estimation provides for great and relevant challenges across all ages and ability range, and it is an absolutely essential (but grossly undervalued) life skill for all of us

Is that DVD player I see in Jakarta for 1.2 million Rupiah a good deal? How many cakes should we bake for the school's charity drive? How much money will I save over the next 20 years if I decide not to smoke? How much is that holiday to Phuket going to cost us, really? Is it worthwhile spending some time to find the best exchange rate? Which handphone package is the best for me? Why is it a waste of time to stay in a long queue to save 5% on petrol? Should I buy a house or should I rent one? I saw this advertisement that tells me we lose about 100 hairs a day! Is that a problem? If I wouldn't produce new hair, how long would it be before I'd be bald?

These are examples of the many numerical questions people face in their daily lives. Yet few of us would stop to figure out the answer— while all it takes is just a pen and a piece of paper. Instead, we go by intuition, trial and error, or we ask others. In daily business decisions, the situation is no better:

My uncle says that a good hawker stall can earn up to \$ 20,000 a month. Is that possible? Shall I make my own website or do I pay a professional \$ 1000 to do it? How should I decide? My three colleagues say they will take 6000 company brochures on their flight to the fair in Vietnam. Why is this not possible, and

did they ever stop to think why they want to take 6000 in the first place? And on what basis did they decide to make 30,000 brochures altogether?

Students like such questions, and teachers can use them to implement and promote some essential teaching strategies:

- questions can be tailored to the individual interests of your students – propose your favorite business to your classmates, and show us why we should invest with you on the basis of a clearly argued income-versus-cost model.
- teachers can promote meaningful team work – estimation questions are often best debated in small groups.
- students will see for themselves that there are many different ways of arriving at the same answer – in my classes I've heard at least 10 different methods of correctly estimating the number of hairs on people's head
- it drives home to students how important clear mathematical communication is – only those students with a clearly argued case will convince their peers.
- it allows for great oral / whiteboard presentations, and thus enhances presentation skills – ask students to come forward to explain how they arrived at their answers.
- it offers a fantastic way to develop research skills – students can go on the internet to verify their estimates, e.g to find out how many hairs people have on their head
- it always turns up some great surprises – some of the students who usually struggle with mathematics can suddenly turn into the class very best, offering teachers a prime chance to motivate such students and bring them back in the fold.
- it empowers students and thus gives them a confidence that immediately translates into better performance on the more traditional maths tasks – seeing that you worked out the correct number of hairs on your head to within 10% is a truly thrilling experience for a student generation with an ingrained belief that they cannot find out things for themselves
- it teaches the students that most important numbers in life are estimations, and that calculators are only good for accountants or as a tool to verify earlier estimations – hence Wheeler's statement at the beginning of this essay.
- it forces the students to employ critical thinking, as they must critically evaluate their own and other's arguments and presentations
- these problems alert students to the holistic nature of knowledge – they have to use insights from many different observations and fields of knowledge.

More generally, the last point can be expanded upon to show that mathematics is, indeed, a vital basic skill that underlies much of our knowledge. We always tell our students that this is so, but many students are still at a loss to explain the use of



mathematics in their daily lives. In my experience, students are truly turned on by societal questions that are relevant to their own lives, beginning with easy fun questions:

- How many chickens do we import every day into Singapore? How big a building would we need to breed all the chickens that we eat? Is Takashimaya big enough?
- If we fill the whole class room with coke cans and each of us can take one drink every day, how long can we drink?
- If we seal the whole room hermetically, how long can we breathe before the air runs out?
- How much money does the McDonalds on the corner make? And your favorite disco?
- If all 6 billion people in the world would come to Singapore, would there be enough room for all to stand?
- How much money is the Singapore government earning out of cigarette taxes?
- And how much do they gain from GST and alcohol taxes?
- (class project) Singapore's total annual budget is around 25 billion. Can you account for how the government receives that amount? (This is a wonderful exercise necessitating much ingenuity and internet research. Filling the gaps will lead the students to truly interesting macro-economic insights about their own country)
- The world raised a billion dollars for the tsunami victims in Aceh. How much is that, really? – translate this figure to a scale on which we can actually comprehend its impact.
- Singapore says it has a water problem. Do you understand why? (Nothing drives home the world-wide looming water disaster as forcefully as finding out for yourself.)
- If the polar caps melt, how far would the sea-level rise? How come your estimates are different from the official estimates you've found on the internet?
- There are 6 billion people in the world right now. How many babies are born every second?
- Why are economists talking nonsense when they say that next year's economic growth will be 3.4%? What would be a more valid way of stating such claims?
- Based on initial data on the SARS disease in Singapore, how rapidly would the disease have spread had the government not acted?

Students are often so transfixed on calculators that they do not realize that most truly important numbers in life are not exact, but are based on estimations. Indeed, the list of questions above can all be initially attempted in class with the help of pencil, paper, and brainpower alone, using the internet to refine key assumptions.

As is clear from the list of examples, the possibilities are endless: teachers could simply pick up the newspaper to find a new problem every day. If the problems are well chosen – preferably connected to areas of personal or community concern – students are very interested in discussing the outcome and each other's findings. By checking their assumptions and estimates versus the

'real' data found on the internet or otherwise, they automatically enhance their research skills. They feel elated and empowered when they notice how often they get the right answer; and they can get truly dug in trying to understand the occasions when they don't. This process of verifying assumptions, checking the consequences and modifying the original assumptions, if necessary, is, of course, the essence of science, and has tremendous universal value. In my experience as a teacher of both physics and mathematics, the question format above is ideal for teaching the scientific method. By listening to each other's contributions, students will learn that there are many different ways of arriving at the same answers; and that the more independent ways we have of arriving at the same answer, the more reliable our estimate is. This is also one area where teachers will learn as much as students: I often had to completely adjust my initial hunch on the basis of my students' careful analysis. Students realize this very quickly and the knowledge that they will teach both peers and teachers alike again empowers them and motivates them to put in their best efforts.

The 'purely' mathematical concept students learn here is the value of logic. List your assumptions. Explain clearly how you derive conclusions from them. If the final answer is wrong, one or more of your assumptions must be wrong. Which is it and why? Refine and try again. Another valuable skill picked up automatically in the process is that of (mathematical) communication. Students will very quickly catch on to the fact that if they wish to convince their peers of their position, the only way forward is a logical and lucid presentation of their assumptions, derivations, calculations and conclusions.

Mathematics lends itself most naturally to address questions on estimation and it would be great to see it as an explicit syllabus topic rather than the implicit one it is now. In the meantime, however, teachers do not need to be afraid to spend time on such non-examined material: the time 'lost' on discussing estimation issues has more than made up by students' vastly increased mathematical confidence, ingenuity, and communication skills.

*Dr Marc van Loo is a cosmologist by training, has worked as a teacher and lecturer across all age groups including adults, and is currently Coordinator of Critical Thinking at NTU. He is the founder/owner of LooLa Adventure Resort, an experiential learning resort in Bintan, and the chief editor of the first independent guide on the IB Diploma, published by Cambridge Press University. He can be contacted at [marc@pacific.net.sg](mailto:marc@pacific.net.sg)*



# Indistinguishable versus Distinguishable Objects in Combinatorics and Probability

Toh Tin-Lam, National Institute of Education

In the topic of Permutation and Combination in the current Additional mathematics syllabus, the concept of arranging or selecting indistinguishable objects versus distinguishable objects is not required. However, from my knowledge, some teachers introduce the problems dealing with indistinguishable objects to their students. In this article, I would like to highlight some issues that might arise from considering indistinguishable objects, and the discrepancy that might arise when this topic is linked to probability.

## An example of Distinguishable Object versus Indistinguishable Object

First, I would like to mention that sometimes it is not a clear-cut case to decide whether a question involves objects which are distinguishable or indistinguishable. Consider the following question, adapted from a question contributed by a teacher attending one of the in-service workshops on Permutation and Combination:

*Trains leave Station A for Station B at every two hour interval starting at 6am and ending at 10pm. The first train leaves at 6 am and the last train leaves at 10 pm. There are two First (F) Class Trains, three Business (B) Class Trains and four Economical (E) Class Trains, each train leaving Station A in two-hour interval. How many ways can the schedule be made?*

From the above question, it is likely that all the nine trains are "distinct". Any two First Class Trains, for instance, may not be exactly identical in terms of the level of comfort they offer, the types of service they have, and the people working on the two trains are different. If you perceive the problem in this way, then all the nine trains are distinguishable. Then logically the number of different schedules refers to the number of different ways of arranging the 9 distinct trains, which gives the answer of 9! ways.

However, from daily life experience, people do not normally see any two trains of the same type as "distinct" as the prices are the same. One will not specifically go for a particular schedule of train because of the difference of the service within the same class! Thus, in practice, from passengers' point of view, any trains from the same class make no difference as the train fare and services are the same. Thus, passengers will see them as indistinguishable rather than distinguishable. Perceiving from this point of view, the number of different ways of scheduling the 9 trains =  ${}^9C_4 \times {}^5C_3 \times {}^2C_2$ .

Thus, in a question involving permutation and combination, sometimes it needs some interpretation before one can decide whether the objects are distinguishable or not. It deserves teachers' attention when they set questions to their students which are of this nature.

## Introduction to the concept of Permutation of indistinguishable objects

Consider the permutation of the four distinct letters A,B,C and D. It is easy to see that there are  $4! = 24$  ways of arranging the four letters. All the possible cases are shown below.

A	B	C	D
A	B	D	C
A	C	B	D
A	C	D	B
A	D	B	C
A	D	C	B
B	A	C	D
B	A	D	C
B	C	A	D
B	C	D	A
B	D	A	C
B	D	C	A
C	A	B	D
C	A	D	B
C	B	A	D
C	B	D	A
C	D	A	B
C	D	B	A
D	A	B	C
D	A	C	B
D	B	C	A
D	B	A	C
D	C	A	B
D	C	B	A

Consider the permutation of the four letters  $A_1, A_2, C, D$ . First we assume that the two A's are distinct. The total number of permutation is still 24 as shown in the following table.

$A_1$	$A_2$	C	D
$A_1$	$A_2$	D	C
$A_1$	C	$A_2$	D
$A_1$	C	D	$A_2$
$A_1$	D	$A_2$	C
$A_1$	D	C	$A_2$
$A_2$	$A_1$	C	D
$A_2$	$A_1$	D	C
$A_2$	C	$A_1$	D
$A_2$	C	D	$A_1$
$A_2$	D	$A_1$	C
$A_2$	D	C	$A_1$
C	$A_1$	$A_2$	D
C	$A_2$	$A_1$	D
C	$A_1$	D	$A_2$
C	$A_2$	D	$A_1$
C	D	$A_2$	$A_1$
C	D	$A_1$	$A_2$
D	$A_1$	$A_2$	C
D	$A_2$	$A_1$	C
D	$A_1$	C	$A_2$
D	$A_2$	C	$A_1$
D	C	$A_1$	$A_2$
D	C	$A_2$	$A_1$

Suppose now we assume that the two A's are indistinguishable. We drop the labels of subscripts from the above table, so that we obtain

A	A	C	D
A	A	D	C
A	C	A	D
A	C	D	A
A	D	A	C
A	D	C	A
A	A	C	D
A	A	D	C
A	C	A	D
A	C	A	A
A	D	A	C
A	D	C	A
C	A	A	D
C	A	A	D
C	A	D	A
C	A	D	A
C	D	A	A
C	D	A	A
D	A	A	C
D	A	A	C
D	A	C	A
D	A	C	A
D	C	A	A
D	C	A	A

Thus it can be seen that when the subscripts are removed from the A's, each case of arrangement of the four letters occurs twice. Hence the total number of permutation of two A's, one C and one D is  $\frac{24}{2!} = 12$  ways, as illustrated in the following diagram where the replica of each permutation is removed. Notice that each of the two permutations that occur twice is removed once to form the following table.

A	A	C	D
A	C	A	D
A	C	D	A
A	D	A	C
A	D	C	A
C	A	A	D
C	A	D	A
C	D	A	A
D	A	A	C
D	A	C	A
D	C	A	A

The division by 2! occurs to remove the cases of repetition and multiplicity which the naked eye cannot see any difference between the indistinguishable letters. Thus the permutation of objects consisting of some indistinguishable objects becomes slightly more complicated. This may not seem to pose any



major problems when we consider arrangement or selection of objects, but the problem may arise when we consider probability, which is a topic in Mathematics Syllabus D. The example below illustrates this fact.

### One Example: Tossing of two indistinguishable coins

Consider the case of two given identical coins (which one cannot tell the difference between the two coins at all). All the possible outcomes are (1) two heads; (2) one head and one tail; and (3) two tails. Since the two coins are indistinguishable, we cannot identify which is the “first” coin and which is the “second” coin. Hence we only have three possible outcomes in total as the three cases above.

Consider the following probability question:

*You are given two identical coins. What is the probability that you will obtain exactly one head and one coin?*

One solution (which is incorrect!) that some students might offer is:

Since there are three possible outcomes, and obtaining one head and one tail is one possible outcome out of three, required probability =  $\frac{1}{3}$ .\*

By using a simulation of throwing two fair coins 1000 times using a probability software twice, the number of heads and their respective frequencies are shown below.

0	247
1	499
2	254
-----	
1000	
sample mean = 1.007	
sample st dev = 0.708	

0	251
1	515
2	234
-----	
1000	
sample mean = 0.983	
sample st dev = 0.696	

It can be seen that approximately 50% of the time one head occurs out of the throws of the two coins. Thus, most likely the answer  $\frac{1}{3}$  is not correct. Let us review the definition of probability again.

Consider the set of all possible outcomes denoted by  $\xi$ , assuming that all the outcomes in  $\xi$  are equally likely to occur, and let A be a subset of  $\xi$ . Then P(A) can be defined as  $\frac{n(A)}{n(\xi)}$ .

When we apply this formula to this situation of throwing two coins, then the set  $\xi = \{\text{two heads, one head and one tail, two tails}\}$ , and  $A = \{\text{one head and one tail}\}$ . One may be tempted to apply the above formula for probability and erroneously obtain the answer  $\frac{1}{3}$ .

However, consider the case when the two coins are distinguishable. Then  $\xi = \{\text{HH, HT, TH, TT}\}$ . The set A, being the event that one head and one tail turn up, is given by  $\{\text{HT, TH}\}$ . Hence probability of obtaining one head and one tail  $\frac{2}{4} = \frac{1}{2}$ .

For the case when the two coins are indistinguishable, the set A collapses into one element, i.e.  $A = \{\text{one head, one tail}\}$ .

However, in the set  $\xi = \{\text{two heads, two tails, one head and one tail}\}$ , the three outcomes are **NOT** equally likely to occur, as the case of one head and one tail occurring is twice as likely to occur as each of the other case. Thus the definition of  $P(A) = \frac{n(A)}{n(\xi)}$  cannot be applied directly to this case without considering the multiplicity of the case of “one head, one tail”.

One way to overcome this problem is that we assume that the coins are *distinguishable*, and solve the problems as if the two coins are distinguishable so that we can obtain the answer  $\frac{1}{2}$ .

Thus, the correct answer to this question can be obtained by “treating” the two coins as “distinct” and labelling them as the first coin and the second coin. Such difficulty in considering probability will not arise when we treat all objects as distinguishable. It is understandable why in O Levels the permutation and combination of indistinguishable objects is not included as it will cause further confusion for the students when they learn probability later on in their Mathematics Syllabus D.

### Introduction to the concept of selecting indistinguishable objects

Consider the following question: I have five red balls, three white balls and two orange balls. All the balls are identical except for their colours (that is, one cannot tell the difference between any two red balls, any two white balls or any two orange balls). How many selections of two red balls, two white balls and one orange ball are there?

Reading the above question and applying the knowledge of combinatorics, one might be tempted to conclude that the number of ways of selecting two red balls, two white balls and one orange ball from a total of five red balls, three white balls and two orange balls is  ${}^5C_2 \times {}^3C_2 \times {}^2C_1 = 10 \times 3 \times 2 = 60^*$ .

However, upon carefully consideration, it is stated that one cannot tell the difference between any two balls of the same colour. In other words, one cannot tell the difference at all except for their colour. In fact, logically the answer to this question is 1 (there is only one such selection).

Consider a related probability question:

*A bag contains five red balls, three white balls and two orange balls. All the balls are identical except for their colours (that is, one cannot tell the difference between any two red balls, any two white balls or any two orange balls). I select five balls from the bag randomly without replacement. What is the probability of me obtaining two red balls, two white balls and one orange ball?*

Let us look at one (incorrect) way of solving this problem. Since all the balls are indistinguishable except for colours, all the possible outcomes of selecting five balls consist of the following:

- {5 Red}
- {4 Red, 1 White}
- {4 Red, 1 Orange}
- {3 Red, 2 White}
- {3 Red, 2 Orange}
- {3 Red, 1 White, 1 Orange}
- {2 Red, 3 White}
- {2 Red, 2 White, 1 Orange}

9. {2 Red, 1 White, 1 Orange} 10. {1 Red, 3 White, 1 Orange}  
 11. {1 Red, 2 White, 2 Orange} 12. {3 White, 2 Orange}

Since there are twelve possible outcomes, and selecting two red balls, two white balls and one orange ball is one of the possible outcome, the required probability =  $\frac{1}{12}$ .\*

However, notice that all the twelve events listed above are not all equally likely. For example, consider the event of {5 Red} and {4 Red, 1 White}. Since the white ball can be any of the three white balls and the four red balls can be from any of the five red balls, the event {5 Red, 1 White} is  ${}^5C_4 \times {}^3C_1 = 15$  times more likely compared to choosing all the five red balls from the bag. Similarly, the event of obtaining two red balls, two white balls and one orange ball is 60 times more likely than choosing all the five red balls from the bag. The twelve events listed above are not equally likely, thus the argument leading to a probability of  $\frac{1}{12}$  in the preceding paragraph is not correct.

The correct answer to the above problem can be obtained by "treating" all the balls as distinct. Hence required probability  $\frac{{}^5C_2 \times {}^3C_2 \times {}^2C_1}{{}^{10}C_5} = \frac{1}{12}$ . For readers interested to read up more on this type of distribution (known as the *hypergeometric distribution*), more can be found from any of Advanced Level Probability

textbook from the chapters involving Discrete Random Variables.

From the above cases of the tossing of coins and the selection of balls, it can be seen that while in concerning permutation and combination, indistinguishable objects do indeed reduce the number of possible choices; while considering probability, all these cases of repetition due to indistinguishable objects actually expand to render each of the observable outcome having a different chance of occurring as the other observable outcome. Thus, the way to overcome this problem in solving probability problem is to consider all the objects as distinguishable. This explains why the problems of indistinguishable objects were not there to confuse students in probability questions but can be a potential source of difficulty in permutation and combination question.

### Conclusion

Hence it is of paramount importance that teachers who intend to teach the concept of selecting or arranging indistinguishable objects, understand the likely misconception that students may have due to the inherent nature of the topic. They must also gauge if students are cognitively ready to grapple with the issues and overcoming their learning difficulties to arrive at the right answer.

## Mathematically Rich Tasks that Promote Thinking and Creativity

Jaguthsing Dindyal, National Institute of Education



The use of mathematically rich tasks that promote thinking and creativity was the main theme of a workshop by led Prof P. Sullivan\* and entitled "The Challenge of Teaching Mathematical Thinking, Creativity and Problem Solving in Mixed Ability Classes" at the Singapore Polytechnic on September 6, 2005. The workshop was jointly organized by The Association of Mathematics Educators (AME) and the Department of Mathematics and Science of Singapore Polytechnic and attended by more than 160 teachers from the primary to JC levels.

Essentially, Prof. Sullivan advocated the use open-ended tasks

to engage students of all abilities in mathematical thinking and problem solving. He defined an open-ended task as one that (1) requires more than remembering, (2) allows students to learn by answering the question, and (3) has several acceptable answers. Here are some of the tasks that he used:

#### Task 1: A million what?

Think of one million of something. Make up an interesting question about your one million something. Answer your question. Be prepared to explain your question and answer to someone else.

#### Task 2: F1 Race track

Draw a Formula 1 race track around Singapore Polytechnic and draw a graph of speed against time that the cars take to make one circuit.

#### Task 3: Open-top box

It is possible to make an open top box by cutting squares from the corners of a rectangular card and then folding up the sides. If the card is 20 cm x 10 cm, what might be the volumes of some boxes that you can make?

#### Task 4: Dimensions of box

The surface area of a rectangular box (rectangular prism) is 94 cm<sup>2</sup>. What might be the dimensions of the box?



### Task 5: Integrand

What might be  $f(x)$ , if  $\int_2^4 f(x)dx = 10$

It can be noted that the above tasks require more than the recall of a basic fact or the use of a standard procedure or algorithm. Students are challenged to think more deeply about the mathematics that is involved. Prof Sullivan claimed that open-ended tasks:

- focus attention onto principles;
- allow students to investigate, make decisions, generalize, seek patterns and connections, communicate, discuss, and identify alternatives;
- allow students more opportunities for creative thinking; and
- address content explicitly and so make it clear to students what is intended that they learn.

Many students are conditioned by the traditional nature of mathematical problems where only one right answer is acceptable. This constrains students not to think outside the box. The open-ended tasks allow students to learn by doing rather than listening. This implies that the learning trajectory is not fixed and rigid but is flexible and takes different turns, allowing for more creative thinking. Students of all abilities get the opportunity to build stronger social relations through their interactions while working on the tasks. Also, all students get the opportunity to work on mathematically rich tasks.

One important question for all teachers is: how do we create open-ended tasks? For making open-ended tasks, Prof. Sullivan suggested two methods:

#### Method 1

- Write down a question and work out the answer.
- Make up a new question that includes the answer as part of the question.

For example, using this method we can ask the question: Write down 1.29 to 1 decimal place. This question has only one specific answer. To make the question open-ended, we can ask: What number when rounded to one decimal place is 1.3?

#### Method 2

- Write down a question and a complete answer to the question.
- Remove some of the question parts to make the question open.

For example, using this method we can ask the question: The base of a cuboid is a rectangle having dimensions 20 cm 30 cm. If the height of the cuboid is 15 cm, find its volume. The answer to this question is = 9000 cubic centimetres. To make the question open-ended, we can ask: Find the dimensions of the base of the cuboid if its height is 15 cm and the volume is 9000 cubic centimeters. In the solution we removed the part pertaining to the dimensions of the base to make an open-ended task.

Another relevant question for teachers is: how do we integrate open-ended tasks into our daily lessons? One strategy may be

the following:

- The whole class works on the same or related tasks.
- The teacher reacts to the students' needs (rather than is proactive)
- (Pre-planned) variations are provided by the teacher based on the students' responses.
- There is a whole-class discussion of selected responses after the activity. The class sees the activity as both an individual and a group goal.

Prof. Sullivan acknowledges that there are some issues about the implementation of open-ended tasks. For example, a context may be unfamiliar or perhaps not interesting for some students; or the mathematical demand of the task may not be suitable for some students. Prof. Sullivan also stated that sometimes it may not be clear which aspects may be contributing to a particular student's difficulty, but we can anticipate some of the factors and prepare prompts to help the student. We can reduce the number of steps involved, simplify the modes of representing results, reduce the degree of abstraction or visualisation required by making the task more concrete, or even just reduce the size of the numbers to be manipulated. Overall, the workshop was well received by most teachers.

*\*Prof. P. Sullivan is Pro Vice Chancellor of La Trobe University at Bendigo, Australia. He is an experienced teacher and researcher with a special interest in mathematical tasks that promote thinking and problem solving. He is the editor of the prestigious Journal of Mathematics Teacher Education, is an author of a mathematics text series and a popular resource for mathematics teachers, and is a member of the Australian Research Council College of Experts.*



## More Open-Ended Tasks

1. What are some fractions smaller than  $\frac{2}{3}$ ?
2. What are the two numbers, if their product is 3.5?
3. Write 10 fractions between  $\frac{1}{3}$  and  $\frac{2}{3}$ .
4. Explain to one of your friends how you would work out which is larger  $\frac{3}{4}$  or  $\frac{3001}{4001}$ .

# In the Beginning Triangles, triangles and more triangles ....

*Carmen Hoo, Raffles Girls' School (Secondary)*

Big? Small? It matters not.  
What is oft more important is that  
Thou art the same in more ways than thou art nought.  
For reasons such as <sup>1</sup>SSS and AAA,  
SAS and SSA,  
ASA and AAS,  
These 3-sided figures art same in all ways save size.  
In fact, all might congruent be, save two: SSA and AAA  
The former be neither congruent nor similar  
and the latter only similar be.

## Trouble Abrewing

Away, away from those 3-sided things  
Doth thy rules too apply to 4-sided ones,  
Figures we know so well?  
Squares: Ah! Simple for all art similar, non?  
For all angles are right and all sides proportional.  
Rectangles: Squares the story the same  
If 2 'neighbours' art long as their partners short.  
Rhombuses too similar if 1 angle is equal  
to a 'friend' in another.  
Parallelograms and trapeziums, similar they are  
If sides adjoined at 1 point  
Art twice as long as their mates in another,  
Perhaps, thrice and merely half.  
and angles betwixt are same in value and place.

The pattern is not unfamiliar.  
Friends, thou art can tell.  
But 'tis not neat like our 3-sided pals where 3 conditions  
suffice

For 4-sided ones, angles and sides,  
Sides and angles. What order?  
How many? Bethink 4? Or even 5?

## The Battle

With 4, I've tried 16 'fiends' in all.  
Not at all enough to tell similarity by.  
Like <sup>2</sup>SSSS and SSSA,  
SSAS and SSAA,  
Figures of 4 sides not at all similar,  
Though mistaken I may be,  
pray tell if that be so.

With 5, success more certain indeed!  
SASAS has been good with sides proportional and angles  
equal

maketh quadrilaterals sameth.  
Alas! Such combinations cumbersome  
for many variations make burdensome  
such pattern, non?

So, my friends, let us start from the beginning again  
with our 3-sided allies where AAA suffices  
but with quadrilaterals, AAAA does not  
for squares and rectangles that condition  
share

'though similar they art not for  
sure.  
Another condition essential  
still.

## (Footnotes)

<sup>1</sup> These are conditions to determine similarity and/or congruency of triangles. SSS indicates all 3 corresponding lengths of sides of 2 triangles are equal in length. AAA implies that all 3 corresponding angles are equal. While SAS condition has 2 corresponding lengths of sides and the included angle are equal, SSA condition is similar to SAS except that the angle that is equal is any one other than the included one. For both ASA and AAS, there are 2 corresponding angles that are equal and 1 corresponding side that is equal in length. Of the 6, SSS, SAS, AAS and ASA are conditions of congruency while AAA is sufficient for similarity. Because of the possibility of ambiguous triangles arising from SSA, this condition is not enough to judge either similarity or congruency.

<sup>2</sup> Labels such as SSSS are taken in specific order (either clockwise or anticlockwise) so that SSAS implies that 2 consecutive sides are equal or proportional in length and the angle made by the last 2 stipulated sides is equal in both quadrilaterals being considered as illustrated in the pair of quadrilaterals below.

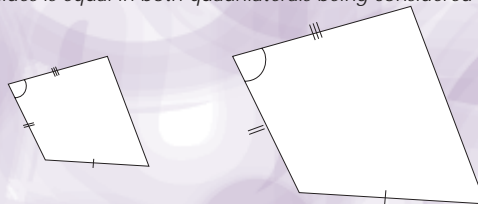


Fig 1

However, 4 conditions is insufficient for similarity of quadrilaterals to be determined as can be seen from this example in Fig. 2 where even though SSAS condition prevails, the 2 quadrilaterals may not be similar since there are 2 possible quadrilaterals with the same dimensions.

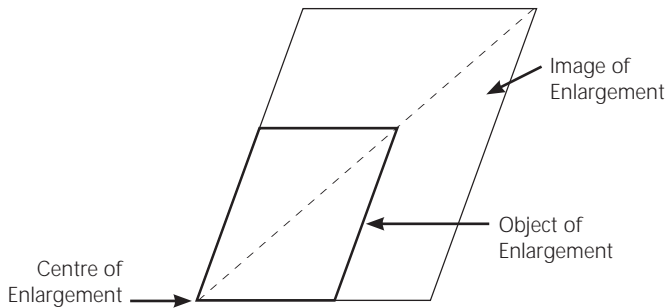


Fig 2



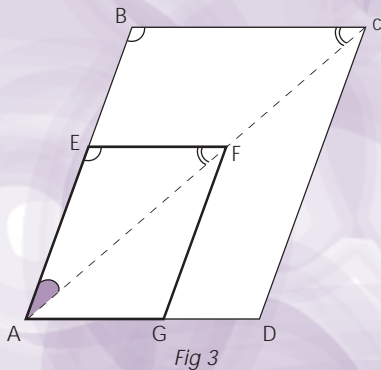
Ah! Similar quadrilaterals are but partners in enlargement  
 – object and its image –  
 When an angle and its corresponding sit one on the other,  
 Vertex on vertex, point to point.  
 This vertex be centre from whence  
 Springs image from object.

Comprehend not these ramblings?  
 Much better in thy mind's eye  
 Will this illustration paint the picture I have in mine.  
 \*3Two vertices facing one to the other  
 Must both on the diagonal lie.  
 This line which from the Centre draw.



This argument the reverse be also true.  
 When thus positioned and all corresponding angles equal,  
 Ay, AAAA yet more essential,  
 If on the diagonal arising  
 From the heart of enlargement aligns  
 The 2 opposite corresponding vertices,  
 Then, such triumph! any 2 4-sided figures must similar be!

Wherefore art triumph arise?  
 Turn thy eyes upon that which is named Figure 3.  
 If all corresponding angles are equal,  
 Then  $\angle ABC$  and  $\angle AEF$  must also profess equality  
 And therefore, BC and EF must, as Euclid recommends,  
 Only intersect at infinity.



Flie on the diagonal AC,  
 equal must be  $\angle ACB$  and  $\angle AFE$   
 for corresponding they must be  
 if CB be parallel to FE.

The angles, their sum always the same  
 In figures with 5-sides, 4 or e'en 3.  
 Thus,  $\angle EAF$  cannot be but equal to its other  $\angle BAC$ .

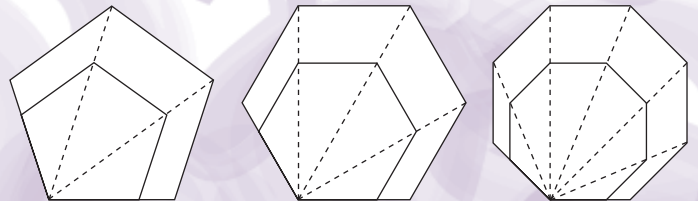
By AAA,  $\triangle AEF$  and  $\triangle ABC$  must be same in all save size.  
 So too  $\triangle AFG$  and  $\triangle ACD$ .  
 No different in reason than their 'twins' afore.

Ah! How sweet success doth taste,  
 for AEF and ABCD must similarity profess.  
 The long, the short for quads to be:  
 When all angles are equal in value and place  
 And 2 vertices opposing to the centre must be  
 On the diagonal from the centre align,  
 And thus, similarity of any 2 quadrilaterals will be ascertained.

Finally, an elegant one for our 4-sided friends,  
 None too different from our 3-sided allies of old.  
 Dare we tread further to shores yet un-ventured and stories  
 yet untold?

And Yonder?  
 Pentagons, hexagons and yonder.  
 What patterns come to pass  
 When all angles and their partners  
 Equal as should be?

Diagonals thou draw unceasing  
 As sides the number increasing.  
 Each diagonal from an enlargement centre arising  
 2 opposite corresponding vertices encompassing  
 And 2 pairs of similar triangles flanking  
 The diagonal creating.



See the 'paintings' yonder for an image of my rambling  
 And picture where each pair of shapes pointing.  
 As n grow greater in number  
 So too the pairs of 3-sided figures.  
 Each pair similar as from reasons of \*4old.  
 From hence, 'tis easy to tell  
 That for n-gons, when all angles art same in value and place,  
 Pairs of corresponding vertices  
 Opposing the centre of enlargement  
 Must on diagonals drawn fulfill criteria of alignment  
 So that similar they must be,  
 number of sides not withstanding.

(Footnotes)

\*3 The 2 corresponding vertices diagonally opposite the centre of enlargement must be aligned on the diagonal thus drawn from this centre.

\*4 Extending from the argument of similarity of quadrilaterals, the triangles created from the diagonals drawn from a single vertex to the opposite vertices also form pairs of similar triangles.

How many diagonals suffice, thou art wondering?

The relationship arising

As the number of sides multiplying,

The number of diagonals increasing

With a consistency so amazing.

When the number of sides is  $n$ ,

The number of diagonals be mere a simple  $\frac{n(n-3)}{2}$

'Tis most astounding for

If all corresponding angles magnitude neither more nor less

And vertices corresponding sit on diagonals  $(n-3)$  no more

Drawn from the enlargement centre, no less

Then any 2  $n$ -gons will similarity profess.

Beyond dreams, such success!

The only exception be 'ole 3-sided pals

For no diagonals to speak of, thus  $(n-3)$  nil.

Hence, the rule for triangles somewhat simpler

Than that for  $n$ -sided closed figures.

Another Battle?

If angles in same positions remain equal in size,

Corresponding diagonals parallel be,

Doth 2  $n$ -gons similar see?

If need be, then turn one to fit the other.

Nay, no images in mirrors but one true to  $\frac{1}{2}$ another.

Then move further,

So vertices cover.

Then 'tis like the  $\frac{1}{2}$ other.

Ah-ha! For my friends, any two polygons must similar be,

None the need for vertices aligning.

If all corresponding angles equal,

If all corresponding diagonals parallel.

In The End

To borrow from Shakespeare, "The deed is done."

There is no more I condone.

For proven that polygons similar

With angles and diagonals, no other,

So simple the triumph,

No troublesome combination.

Constant 'til the end, so patient are you

Testing you no further, I bid a grateful *adieu*.

(Footnotes)

<sup>5</sup> An investigation of the number of diagonals that can be created from a single vertex yielded two relationships. (1) If all corresponding angles are equal and all corresponding vertices are aligned on diagonals drawn from a single vertex, then to determine similarity, the number of diagonals is  $(n-3)$  for an  $n$ -sided polygon. (2) Of lesser consequence in the validity of similarity is that the number of pairs of similar triangles thus formed will be  $(n-2)$ .

Name of Polygon	Number of sides	Number of diagonals	Number of pairs of triangles
Quadrilateral	4	1	2
Pentagon	5	2	3
Hexagon	6	3	4
Heptagon	7	4	5
Octagon	8	5	6
$n$ -gon	$n$	$(n-3)$	$(n-2)$

Table 1

<sup>6</sup> This proposal only works if the orientation of the polygons are the same and not for mirror images.

<sup>7</sup> This refers to the centre of enlargement explanation.

## Classroom teaching and TEACH LESS LEARN MORE

Teo Soh Wah, National Institute of Education

This article is about aligning the idea of 'Teach Less Learn More' with the 'How' to conduct classroom teaching in relation with the 3Ws - When, What, and Why. I will elaborate on the 3Ws one at a time.

### The first W, 'When'

When the class environment is conducive for learning, classroom teaching will be effective. Classroom teaching can take place only when students are engaged in learning. Below are some possible ways to engage our students at the start of the lesson.

- Capture their attention by asking questions. Before we

teach map reading, or proportion, start with a question like, "How many of your parents drive?", "Have you ever experienced trying to find your way when you are lost?"

- Include life stories of great mathematicians with slides, cartoons or movie clips. This will make our lessons more lively and interesting.
- Create surprises. We can perform magic tricks or show a mathematical fallacy to arouse the students' curiosity so that they are attentive in our lessons. To illustrate, consider the following example:



Given	$a=b$
When we multiply both sides by a, we have	$a^2 = ab$
Then we minus $b^2$ from both sides, we have	$a^2 - b^2 = ab - b^2$
Now factorise both sides,	$(a + b)(a - b) = b(a - b)$
Finally cancel $(a - b)$ ,	
	$a + b = b$
Let $a = b = 1$ ,	$2 = 1$
	Why?

After engaging the students' attention at the start of a lesson, we have to ensure that the classroom environment is conducive for effective learning to take place. Cooperative learning will help students who derived a sense of achievement in learning and greater satisfaction by working with their friends. Also a good learning environment that is caring and supportive will help to create positive peer influence among the students. For example, the foreign scholar who is strong in mathematics may be paired with the local student who is weak in the subject. The key idea is to leverage on the students' strengths and work on their weaknesses.

Teachers don't just teach, they create an environment for learning. Work with students to create and improve the classroom learning climate. "The best teachers may never "teach" anything at all. In contrast, they create environments in which vast amounts of learning may take place." — S.I.Hayakawa.

The "teach" is in quote, because teaching does not imply spoon-feeding but facilitating learning, guiding and most of all, leading the students to think.

The following three teaching scenarios depict the different responses of the teacher to the mistake made by the student.

A student was solving a quadratic equation:  $x^2 - 3x - 4 = 6$   
 He immediately went on to factorize the left hand side of the equation, and obtained  $(x-4)(x+1)=6$   
 By applying his usual way of solving equations which is equating the factors, he obtained  $(x-4)=6$  ,  $(x+1) = 6$  and acquired the answers:  $x=10$  ,  $x =5$   
 Then he checked his answers, he found that they were wrong. So he asked his teacher:-

**First scenario**

Teacher 1: Why do you always make this type of mistake? I have told you before that you have to arrange the equation till it is equal to zero. You never listen!!  
 Re-do!

The student felt sad and the teacher was angry..

**Second scenario**

Teacher 2: Well, your second step is wrong. You should arrange the equation till it is equal to zero first.  
 Student: But how?  
 Teacher 2: You have to ... (Teacher 2 patiently explains the solution to the student)

The student was happy because he understood his mistake and the teacher experienced a sense of achievement.

**Third scenario**

Teacher 3: Can you explain to me how you get 10 and 5?  
 Student: Because both the factors are equal to 6!  
 Teacher 3: Do you mean  $6 \times 6 = 6$ ?

The teacher's question has provoked his student to think and to discover his own mistake. After a while, the student suddenly exclaimed, "Oh... no, only when the product is zero, I can let either factor be zero."

The teacher and the student both felt extremely happy and experienced a great sense of achievement.

All three teachers managed to achieve their objective - getting their students to solve the equation correctly but with very different learning outcomes. In the first scenario, with the hostile learning climate, the student may do well for mathematics but may eventually grow to dislike the subject. In the second scenario, the learning climate is positive, the student would do well in mathematics and grow to love the subject. In the third scenario, the learning climate is very nurturing, the student would be inspired, not only to do well for mathematics, grow to love the subject and enjoy doing so.

**The second W, 'What'**

What is classroom teaching? With the concept of 'Teach Less Learn More', our approach should be geared towards being student-centered, with the teacher acting as a facilitator in helping the students acquire the knowledge. In my view, 'Teach Less Learn More' (TLLM) is a calling to all educators to engage the students and prepare them for the test of life rather than confine them to a life of test. This implies that we have to provide greater opportunities for students to develop holistically with greater interactions between teacher and the learner.

The three main points below encapsulates my understanding of TLLM:-

- Encourage the students to learn more actively and independently
- Nurture a curiosity that goes beyond the formal curriculum
- Inspire passion for learning that carries through life

The accent of the slogan, TLLM is basically to involve students in devising their own learning, leverage on their strengths and provoke them to think. Many teachers may be puzzled especially in cases where explanation of concepts is needed, facilitating may not ensure the desirable end-results. The emphasis here is to help the students learn to be accountable for their own learning. When explaining a concept, we can always build in questions to scaffold, hints, and prompts to guide the students to think. The involvement in the development of the concept will in turn invoke a sense of ownership for the knowledge acquired and have a greater impact on the students.

At the same time, the teacher must know the student well enough to spot his strengths and build on it. For example, for students who are computer savvy and like to surf the net, assign them with the task of finding the history of Pythagoras (or a mathematician's life story) and make him share his knowledge with the class. In cases where students are garrulous, provide opportunities for these students to report their findings of a puzzle or a challenging problem in the class. If students are able to acknowledge their own strengths, you will be able to 'Teach Less Learn More' in the class and at same time help them to learn with drive and enthusiasm.

Besides all of the above, the teacher must change his mindset, from a dispenser of knowledge with students as passive learners to a facilitator that provoke thinking by using more student-centred activities. Teacher 3, in the three teaching scenarios is

a good illustration of the above point. If we allow our students to extend their thinking and be responsible for their learning, they will be more engaged during lessons and enjoy what they are doing. Most importantly, they will acquire good habits which are crucial for life-long learning.

Next, let us probe into the greatest and most common problem faced by all students in the learning of mathematics- mistakes made due to carelessness.

Carelessness cannot be avoided but can be reduced to a minimal level by adopting the habit of 'Check, Step-by-step and Simplify' (CSS). Inculcating this good habit of CSS in the students will certainly help to escalate their learning curve. Let me quote the famous Chinese saying, "Give a man a fish you feed them for a day. Teach him how to fish and you feed him for life". I would advise mathematics teachers to put in more effort and thought in the inculcation of this habit, CSS in the students to ensure independently learning.

My second point on TLLM is nurturing a curiosity that goes beyond the formal curriculum. In order to arouse the students' curiosity, one of the ways is to relate the learning of mathematics to real-life situations. For example, when we teach the topic on Ratio and Proportion, include architectural designs and applications of the golden ratio in Artwork. There are many such authentic problems related to our daily life that are found in our local textbooks. Books from the two publishers, Shinglee and Pan Pacific have included practical problems at the start of each chapter. For Additional Mathematics, life stories of Mathematicians, puzzles and quizzes are also included in the chapters. Besides these stories, the internet contains a reservoir of teaching resources that teachers can use to enthrall their students. One such websites is the homepage of Mathematics and Mathematics Education MME, <http://math.nie.edu.sg/>.

Lastly, to inspire the passion for learning that carries through life, teachers will have to teach with enthusiasm and passion. When we love the subject or at least show our enthusiasm in helping our students they will in turn learn to love the subject too. Confucius has put it as 知之者 不如 好之者, 好之者 不如 乐之者. I translated the saying as "Good to master. Better to love. Best to enjoy." I sincerely hope teachers do not only teach so that students master the topic but provide opportunities for them to create, love and enjoy whatever they learnt. William Arthur

Ward also says, "The mediocre teacher tells. The good teacher explains. The superior teacher demonstrates. The great teacher inspires." If we look at the three teaching scenarios again, all the three teachers managed to achieve their objectives but with very different end-results. Teacher 1 tells and reprimands, Teacher 2 explains and Teacher 3 inspires.

### The third W, 'Why'

The third 'W', the underpinning philosophy of 'Why we choose to join the teaching service' that serves as a touchstone in our drive towards providing quality lessons in our classroom instruction. The philosophy determines how we conduct our lessons and evaluate our students' performance. If we teach in order to help our students pass their examination, the focus of our teaching is very exam-oriented and our evaluation is skewed towards academic achievement. However, if we teach to inspire our students to love and enjoy learning the subject, our teaching and evaluation will be geared towards a more holistic approach.

To conclude, I would like to summarise the article in the following statements. All eight strategies included in this short summary are highlighted (underlined>

- When Be Sincere and Enthusiastic in our teaching  
Create a caring, and conducive Environment for our student to learn
- What Know our students well to Match their learning with our teaching styles  
Be flexibility in applying the Theories.  
Inculcate good learning Habits and leverage on the students' Strengths.
- Why Obtain Feedback to improve on our students' learning  
  
Good teachers carry out and accomplish much more than these eight strategies in their classroom instructions.

I hope this article will serve as a trigger for mathematics educators to share their views on TLLM and their implementations of TLLM in the schools. I would like to conclude the article by grouping the underlined letters of the eight key words into my vision for mathematics education in Singapore: **SEE MATHS Flourish**.

## FORTHCOMING EVENTS FOR YOUR ATTENTION!

	Talk / Workshop	Speaker	Audience	Date & Time	Venue
1	The Teaching of Mathematics	Roger Howe	ALL Mathematics Teachers	Jan/Feb 2006	Singapore Polytechnic
2	Mathematics Teachers' Conference	Various speakers	All Mathematics Teachers	Thursday 1 June 2006 9am - 5pm	National Institute of Education

**EDITORIAL COMMITTEE MEMBERS:**

- Mrs Tan-Foo Kum Fong
- Dr Ng Swee Fong

**Contributions Invited**

Please send to:

**The Editor**  
 Maths Buzz c/o Mathematics & Mathematics Education  
 National Institute of Education  
 1 Nanyang Walk Singapore 637616  
 Email: [kffoo@nie.edu.sg](mailto:kffoo@nie.edu.sg)