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Maths Buzz

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Contents

President's Message...

Mathematics Teachers Conference

Theme: Enhancing Mathematical Reasoning

1st June 2006

National Institute of Education

Dear Mathematics Teachers,

Mathematics reasoning is one important aspect of the Process component of the Singapore Mathematics Curriculum. Reasoning to justify mathematical results is also a unique trait that mathematicians develop and use extensively in their work. It is thus important for school students to acquire this as early as possible in their study of mathematics.

This Conference will examine various degrees of rigour about mathematics reasoning and explore different techniques that can be used to help students develop their mathematical reasoning effectively. Participants were join one of the three groups to examine strategies for reasoning in primary, secondary, and Junior College level so that they can focus on expectations at these different levels. Mathematics educators and mathematicians will lead the workshops to discuss ways to enhance students' mathematical reasoning. The conference is also a good opportunity for participants to establish network with fellow teachers who share similar concerns and wish to do something to improve their own practices.

This Conference has attracted more than 800 registered participants. We look forward to meeting you soon.

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Ebony and Ivory: A Discussion on the Arrangements of a Keyboard.

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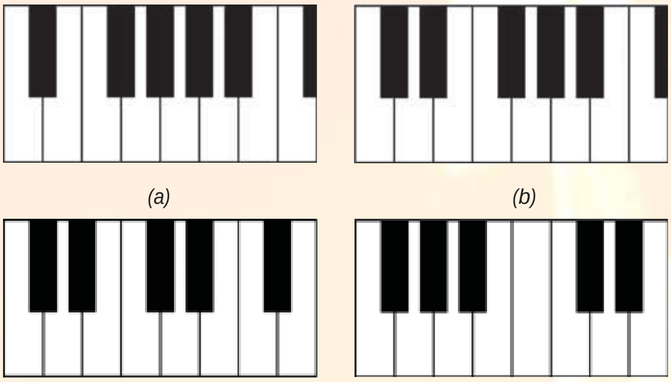
Introduction

In the May 2005 issue of this publication, the temperament of the western musical scale was discussed in the article "*M & M: An exploration of the links between Mathematics and Music*". In this article, we shall explain the reasoning behind the present arrangement of the piano keys as well as the role of mathematics in determining such an arrangement.

The piano is one of the most popular musical instruments in the world. Tracing the genealogy of the piano, it is learned that the modern piano had undergone several changes in terms of the sound it produces as well as the mechanisms involved in producing the sound. Despite these changes, one feature of the piano remains invariant: the arrangement of the keys. Figure 1 presents four possible arrangements of the white and black keys in an octave on a piano keyboard, and readers may be able to correctly identify the one typically seen on the keyboard.



(a) *(b)* *(c)* *(d)*

Figure 1. Possible arrangement of the piano keys

The identification may not be a difficult task even for those who have never learned music because the juxtaposed arrangement of the black and white keys is a common feature in many artistic motifs. But why is the piano keyboard arranged as a set of three white keys with two black keys and another set of four white with three black in an octave? Is such an arrangement based on 'convention' or is there a more substantial reasoning behind selecting this arrangement over the others? The answers to these questions will be revealed in the subsequent section.

How the Present Musical Scale Came About

To understand the present arrangement of the keys, we have to begin from the "just scale", which was simply made up of only the white keys. In our previous article (see Tan & Chua, 2005), we showed how the frequency ratios of other intervals such as from C to D and from C to E can be worked out mathematically using the transitivity approach when the ratios from C to F, to G and to the next C, denoted by C', are known. Table 1 shows the frequency ratios for all the various intervals when computed.

Table 1
Frequency Ratios of the Notes from the "Just Scale" with Reference to C

C to:	D	E	F	G	A	B	C'
Frequency ratio	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$

The same transitivity approach can also be used to subsequently work out the frequency ratios of consecutive notes such as D to E, E to F, and so on. The computed values of these frequency ratios are provided in Table 2.

Table 2
Frequency Ratios Between Consecutive Notes

	C → D	D → E	E → F	F → G	G → A	A → B	B → C'
Frequency ratio	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$

A close examination of Table 2 reveals two noteworthy findings. First, the frequency ratios between consecutive notes take essentially three different values: $\frac{9}{8}$, $\frac{10}{9}$ and $\frac{16}{15}$. Second, only the two intervals from E to F and B to C' have frequency ratios of $\frac{16}{15}$ (=1.07), which is less than 1.10, whereas the other frequency ratios of $\frac{9}{8}$ (=1.13) and $\frac{10}{9}$ (=1.11) are all more than 1.10. At this juncture, readers may wonder what is so important about classifying the intervals into these two broad categories of less than 1.10 and more than 1.10, because after all, all the intervals seem to have the same frequency ratio of 1.1 when rounded off to one decimal place.

However, this uneven distribution of the naturally occurring intervals between notes in the "just scale" caused a dissonance when played together. To mitigate this problem so as to produce harmonious sound, the interval between C and C' had to be divided up in such a way that the consecutive notes had the same frequency ratios. The way to go about doing it comes from a consideration of the frequency ratios in Table 2. The table clearly shows that the intervals with the smallest frequency ratio, the E to F and B to C' intervals, separated the C to C' interval into two sections: C, D and E in one part, and F, G, A and B in the other. To use the smallest frequency ratio as a basis to further divide those intervals with a ratio greater than 1.10 seemed a logical and straightforward action to take to obtain equivalent

(Footnotes)
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frequency ratios for consecutive notes. Consequently, such an action resulted in the insertion of five new notes: Between C and D, D and E, F and G, G and A, and A and B. Thus a totally new musical scale was developed. The effect of this action produced the keyboard arranged as shown in Figure 2.

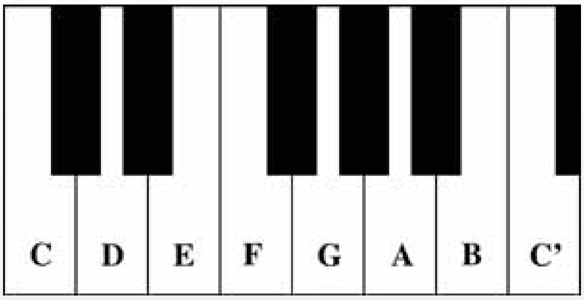

Figure 2. The new musical scale

It should be highlighted that the five newly added keys actually correspond to the black keys in an octave. Also, as clearly seen from Figure 2, two of them are created in the section comprising C, D and E while the remaining three are located in the other section that consists of F, G, A and B.

The frequency ratio of consecutive notes in the new musical scale is $2^{\frac{1}{12}}$, or the twelfth root of 2. Interestingly, the frequency ratios for those intervals listed in Table 2 remain almost the same even under this scale. For instance, the frequency ratio for the interval E to F is $2^{\frac{1}{12}}$, or 1.0595 when evaluated to four decimal places, and this value is very close to $\frac{16}{15}$. Likewise, the frequency ratio for the interval F to G, which is $(2^{\frac{1}{12}})^2$, or 1.1225 when evaluated to four decimal places, also does not differ very much from $\frac{9}{8}$.

In the next Maths Buzz article, we will explain why the frequency ratio between any two consecutive notes of the twelve-tone scale is the twelfth root of 2, and not twelfths.

Reference

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Impact of Math Journal Writing on Students' Performance and Attitude: An Action Research Project at St. Hilda's Primary School

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Introduction

In our school, journal writing is mainly used for the purpose of having students record their personal feelings, somewhat similar to writing a diary. This limited use seems to undermine the pedagogical benefits of journal writing that is touted by many educators.

In view of the shifting trend towards the use of informal modes of assessment, we decided to explore the pedagogical benefits of journal writing in the mathematics classroom. As students tend to memorise the procedures without understanding the reasoning behind the steps, they can show how a problem is solved in sequence but when asked to explain the steps, are usually not able to articulate the concept behind each step. Likewise, students may score perfect marks, but they have little knowledge of the concept behind the algorithm. With this in mind, we asked: *"How can we get our students to reflect and explain the steps they used?"*

We felt that journal writing can offer a way to discover the thought processes of our students and allow us to assess their level of comprehension. Indeed, it is often said that when we write, our thoughts become clearer and we discover what we are actually thinking.

This action research was designed to investigate if the use of math journals in mathematics lessons can increase comprehension of the math topic and students' attitude towards math. We examined three research questions. The first question was whether there is a positive relationship in the test scores for students who engaged in journal writing and those who did not. The second question was whether there are any changes in students' attitudes towards math lessons on days when journal writing is done in class. The final question was whether journal writing has an impact on students' attitude towards mathematics and self-assessment.

Math teachers tend to adopt a direct teaching approach. Learning is seen as a transmission of knowledge from the teacher or text to the students, who reproduce that knowledge. Teachers may also emphasise the "steps" to solve the math question rather than the reasoning for the "steps".

However, the current constructivist approach towards learning describes learning as a cognitive activity that involves an active process of constructing knowledge. This means that whatever gets into the mind has to be constructed by the individual through knowledge discovery. Journal writing is based on the Constructivist Theory of learning and is seen as a tool that can help students construct knowledge. Edwards (1999) explained that the purpose of math journals can be diverse: it can focus students on a review of concepts and can also serve as a knowledge indicator to gain students' views on a topic before its introduction. It can also be used as an assessment tool for teachers to find out how well concepts taught in class have been understood. Hence, the actual writing of a journal entry will assist the student in better understanding the verbal and visual processes. In a recent study, Glover (2005) wrote: "students understand and remember concepts better when they have to convert them into different forms. In doing so, students recognise the ideas as their own and no longer the authors', thus making them easier to remember." Thus, students are more likely to recognise the relevance of mathematics in the real world and be more optimistic in the classrooms. This will provide motivation for both the students and teachers to deal with mathematical confusions.

Methodology

Participants

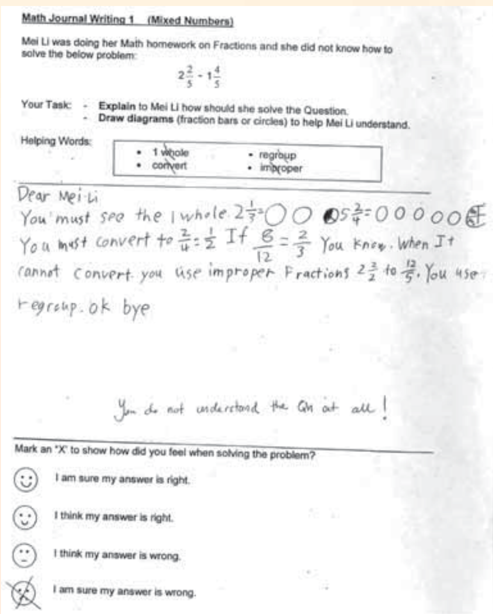
Two Primary 4 classes were selected for the journal writing. Primary 4/2 had 37 students and P 4/4 had 38 students. Both classes were classified as "Average Classes" (with the average entry scores of the students being 52.8% and 53.8% respectively). Primary 4/2 was taught by Teacher A and P 4/4 by Teacher B. A third class (P 4/3) was also selected as the Control Group. Students in this class did not do journal writing. They were taught by Teacher C. The research was conducted over a two-week period from 25 April to 6 May 2005.

Procedure and Date Collection

Two periods (60 minutes) per day were allocated for Math lessons. Typically, teachers would allocate a portion of the instructional time for teaching and the rest for completing the math practice book or related math worksheets.

Before students started writing their journals, Teacher A and Teacher B would explain the journal prompts. Students were then given thirty minutes to complete their journal writing. Three math journal writings were completed. They were related to specific concepts on fractions, and students were required to explain in written form. Both journals were intended to find out how well concepts covered in class were understood. The first journal (Math Journal 1) was administered at the end of Week 1 and the second journal (Math Journal 2) was administered at the end of Week 2.The journal prompts were consistent across both classes (P4/2 and P4/4). All journal writings were completed in the class and collected on the day. On these two journal writing days, teachers observed students' behaviour and expression while they wrote their math journal. Some guiding questions that teachers used were: *"Did students look bored?"; "Did they look stressed?"; "Did they seem to be concentrating?"; "Were they rushing to complete their journal writing?".*

Teachers wrote simple comments on students' writings, such as *"Your journal is out-of-point", "Concept not explained clearly", "Good effort".* The final journal writing (Math Journal 3) was conducted after the two-week period to find out if they felt journalswriting was beneficial. See the two examples below.



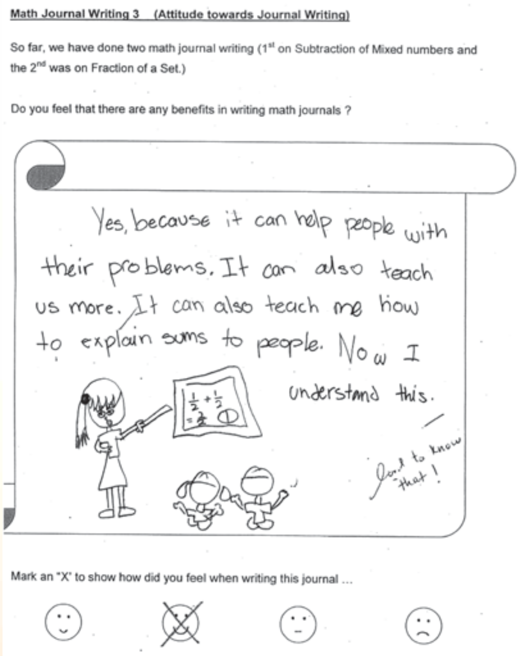


Figure 1. Samples of student journal writing.

To measure achievement, a Pre-Test and Post-Test relating to the topic, Fractions, were administered at the beginning of Week 1 and at the end of Week 2. The students from the journal writing classes sat for both tests, whilst the Control class only sat for the Post-Test. This was a lapse that need to be rectified if this project is to be replicated.

The student attitude was measured daily at the end of every Math lesson using two questions: (a) *I found today's lesson enjoyable.* (b) *I put in effort in today's lesson*. The students rated their responses on a 4-point agree-disagree scale. Their responses were used to monitor changes in reactions on journal writing days versus non-journal writing days.

Data Analysis

Teacher A and Teacher B keyed-in students' scores for the Pre-Test and Post-Test in Microsoft Excel. To avoid any human calculation errors, formulae were set in the Spreadsheet and the test scores were analysed, in terms of averages and difference in scores. Students' responses to the two survey questions over the two-week period were collated using the Optical Analysis Reporting System ("OAS").

Result and Discussion

Cognitive Impact

The table below shows the class average for the Pre-Test and Post-Test scores in the two journal writing classes and the non-journal writing class. The maximum score for each test was 20.

Table 1: Class Average for Pre-Test & Post-Test Scores

		Pre-Test Average (marks)	Post-Test Average (marks)	Post-Test Average - Pre-Test Average marks) or (%)
Journal Writing Class	P 4/2	10.7	14.4	+ 3.7 or 34.6%
	P 4/4	11.2	13.7	+ 2.5 or 22.3 %
	Average for P 4/2 & P 4/4	11.0	13.7	+ 3.1 or 28.4%
Non-Journal Writing Class	P 4/3	NA	11.2	NA

Both journal writing classes recorded an improvement in class average from pre-test to post-test. This may indicate that journal writing had a positive impact on students' comprehension. However, we found it difficult to conclude any direct relationship between the two because of other factors: home tuition and even subtle adjustments made by Teacher A ad Teacher B in their teaching emphasis, subsequent to information gathered after reading students' math journals. Since the non-journal writing class did not take the pre-test, it was not possible to ascertain the extent of improvement for this class.

Affective Impact

The positive attitude was apparent from responses to the self-assessment question: *"How did you feel when solving the question?"* in Math Journal 1 and Math Journal 2, as shown in Table 2.

Table 2: Students' Response to the Self-Assessment Question in the Math Journals

		Math Journal 1 Total Counts for:		Math Journal 2 Total Counts for:	
		Responses 1 and 2	Responses 3 and 4	Responses 1 and 2	Responses 3 and 4
Journal Writing Class	P 4/2	31	5	24	12
	P 4/4	30	8	34	4
Percentage (%)		82%	18%	78%	22%

Foot Note:
- *Response 1: I am sure my answer is right.*
- *Response 2: I think my answer is right.*
- *Response 3: I think my answer is wrong.*
- *Response 4: I am sure my answer is wrong.*

* *Responses 1 and 2 are grouped as positive self-assessment.*
* *Responses 3 & 4 are grouped as negative self-assessment.*

The students expressed positive feelings towards math journal writing. Most of them (89%) felt that writing math journals was beneficial to them. Some common comments were:
- *I like to do math journals*
- *I find math journals interesting.*
- *Please continue doing math journals.*
- *Math journal helps me understand the topic better.*
- *Math journal helps me explain more clearly.*

Although not a part of the original action research, we decided to also examine whether the higher ability students enjoyed journal writing as much as the lower ability students. The students from the best Primary 4 class (P 4/10) were also assigned Math Journal 3. Teacher A also taught this class and the students had written math journals before. The finding was rather consistent, with about 86% stating that journal writing was beneficial, and these students gave positive comments similar to those cited by the lower ability students.

Based on classroom observations, students generally took their journal writing seriously. They did not rush to complete their writing. Instead, many demonstrated considerable thinking when writing their math journals. Our assumption is that students perceived journal writing as a less threatening form of "homework" and was much preferred over the regular type of math homework. This observation is consistent with an idea discussed by Taylor (2004), namely that journal writing seemed to benefit students who were "disorganised". These students frequently forget their homework or have to rummage through their bags to find their homework. Journal writing helps these students pull together their thoughts, as writing in itself requires organisation.

On the other hand, some students either wrote out of point or failed to explain the concepts clearly.

Implications and Conclusion

Our findings did not provide sufficient evidence to suggest that journal writing helps students comprehend a topic better or help them perform better, than the non-journal writing class. But in terms of its impact on the affective element, journal writing does seem to improve students' self-assessment and confidence. This may help students develop a more positive feeling and attitude towards mathematics.

More importantly, through this action research, both Teacher A and Teacher B found that journal writing allowed them to gain insight into the mental processing of their students, an aspect likely to be overlooked in a test situation. This knowledge allows the teachers to reflect and modify their teaching to address any misconceptions of their students. Needless to say, the journal prompts must therefore be very specific and clear, so that the students understand what responses are expected. Through this type of cognitive journal writing, the teachers can assess the student's true understanding of a mathematics concept.

Lastly, we feel that if students develop positive feelings towards mathematics and gain a better understanding of their mathematical thinking, it should lead to better mathematical performance. This matches the central theme of the SAIL approach, which requires students to "describe, use and explicate their thinking process." We feel that journal writing is indeed a suitable platform for students to engage in active and reflective learning.

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Squeeze your brain

A census taker comes to the house of a mathematician and asks how many children he has and what their ages are. The mathematician replies that he has three children and the product of their ages is 72. The census taker replies that he has not been given enough information to determine their ages. The mathematician adds that the sum of their ages is the same as the house number. The census taker leaves but returns in ten minutes and tells the mathematician that he still does not have enough information to solve the problem. The mathematician thinks for a short while and then adds that the oldest child likes ice-cream. The census taker replies that he has enough information and leaves.

Adapted from an unknown source
The answer is found on page 11

Defective Assessment Items in Mathematics

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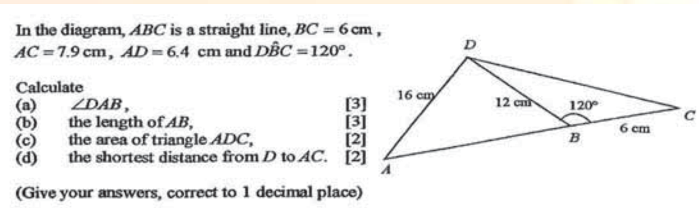
In mathematical instruction, teachers use a large number of problems as test items. Some of the items are copied from textbooks and past examination papers whereas some are modified from similar sources to suit the specific nature of their classes. In addition, a fairly large number of items for tests are originally written by the teachers themselves. While test items from other sources do have a few shortcomings, original test items and modified test items tend to have more. Hence, it is imperative that mathematics teachers take extreme care in writing their test items. In what follows, shortcomings in four test items are highlighted (Items 1, 2, 3, and 4).

Consider the following item (Item 1). While this problem seems to be set nicely as a problem in trigonometry, closer examination reveals some serious flaws. If *BC* = 6 cm then *AB* = 1.9 cm and hence the part of the figure containing triangle *ABD* does not exist, based on the fact that the sum of the lengths any two sides of the triangle should be larger than the third side. The notation for angles *DBC* and *DAB* used in the problem are inconsistent. Furthermore, regarding the amount of work required to be done in the different parts of the problem, a disproportionate number of marks are allocated for the first two parts as compared to the last two parts.

Item 1

Such items are unfortunately quite often set by teachers without


In the diagram, *ABC* is a straight line, *BC* = 6 cm, *AC* = 7.9 cm, *AD* = 6.4 cm and $D\hat{B}C = 120°$.

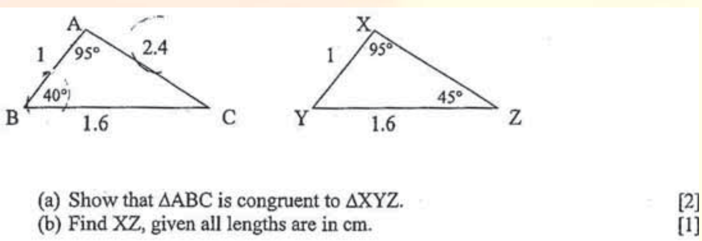
Calculate
(a) ∠*DAB*, [3]
(b) the length of *AB*, [3]
(c) the area of triangle *ADC*, [2]
(d) the shortest distance from *D* to *AC*. [2]

(Give your answers, correct to 1 decimal place)

a proper monitoring system in place.

The next item (Item 2) also has a serious defect. In the writing of this item, the teacher has completely ignored the constraints imposed by the dimensions that he or she has chosen for the angles and lengths of the sides of the triangles. In this case, the longer side is not opposite the larger angle. This may be misleading to students and the figure as whole presents incorrect mathematical information. As the problem is about congruency, many students may still attempt to solve the problem disregarding this very serious flaw in the figure. Furthermore, there is no stem detailing the information given in the figures.

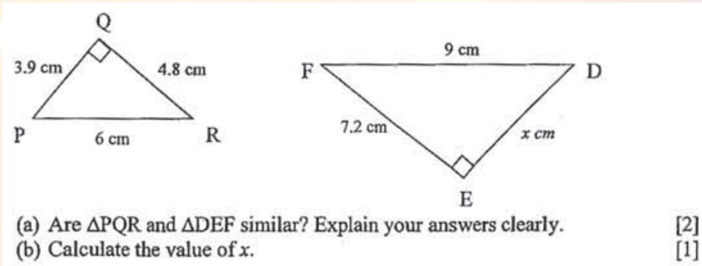
Item 2



(a) Show that ΔABC is congruent to ΔXYZ. [2]
(b) Find XZ, given all lengths are in cm. [1]

The dimensions of the triangle seem to pose a few difficulties to the writers of the test items. In Item 3, a similar flaw, as in the previous two items, can be identified. The writer of this item has provided two right angled triangles which are supposed to be similar. However, in giving the dimensions of triangle *PQR*, the writer has completely ignored Pythagoras theorem which applies to all right angled triangles ($3.9^2 + 4.8^2 \neq 6^2$). Many students might ignore this point and even manage to get a numerical answer based on the ratio of corresponding sides. Again, the writer has not provided a stem detailing the information given in the figures.

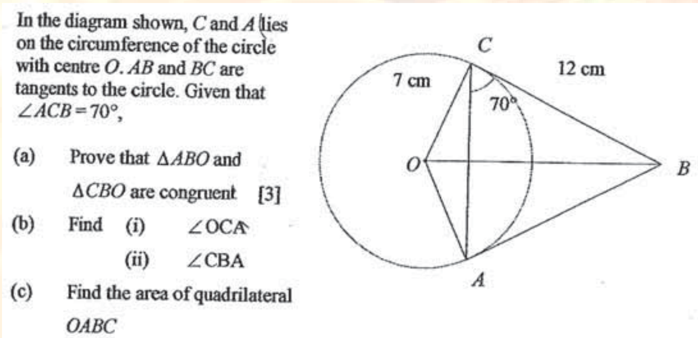
Item 3



(a) Are ΔPQR and ΔDEF similar? Explain your answers clearly. [2]
(b) Calculate the value of *x*. [1]

The idea of inappropriate dimensions can also be noted in the next item (Item 4). The writer of this item has ignored the fact that fixing the dimensions of the right angled triangle fixes the sizes of the angles within the triangle. Based on the given dimensions and the symmetry properties of the circle it is impossible that angle *ACB* = 70°. In this item as well, no stem is provided. Also, no marks are allocated for parts (b) and (c). Besides, there is a grammatical error as well in the first sentence.

Item 4


In the diagram shown, *C* and *A* lies on the circumference of the circle with centre *O*. *AB* and *BC* are tangents to the circle. Given that ∠*ACB* = 70°,

(a) Prove that Δ*ABO* and Δ*CBO* are congruent [3]
(b) Find (i) ∠OCA
(ii) ∠CBA
(c) Find the area of quadrilateral *OABC*

Each of the four items described above can be attempted by students to get an answer that may make some sense within the topic area, despite the inherent flaws in the items. Many students will focus only on getting the answer and ignore the improbable nature of the figures that they have to deal with. The assessment of students' performance based on such items is bound to be difficult to interpret. If the students apply the correct procedures and rules that they would normally apply for a well-designed test item and get the expected numerical answer, do we penalize them for not spotting the flaw in the question? If a student does not get the expected numerical answer, do we give the student the benefit of the doubt that he or she could not attempt the problem because of the inherent flaw in the problem? Thus, defective items in a test can become a very serious issue. It is in the interest of one and all that some proper monitoring mechanism be in place to eliminate such items from school tests.

Teachers may improve the quality of their test items by being more systematic. A few questions that they can ask themselves: What is the purpose of this item? What are the objectives to be tested? What is the mathematical topic or content area on which the item is based? Does the wording of the item correctly convey all necessary information? Are the correct direction verbs used? Is the corresponding figure associated with the item drawn correctly? Are all dimensions in the figure possible within the imposed geometrical constraints? What are the resources to be used in solving this problem? What is the expected answer? Does the answer make sense? Who will work on the items? How much time is to be spent on the solution of this problem? While this list is not exhaustive, it provides some support to the teacher. It is advisable that test items be pilot tested. A fellow colleague can also provide valuable suggestions on how to improve any test item.

Some issues on Partial Fractions

Toh Tin-Lam, National Institute of Education

Introduction

Partial Fractions will be introduced into the new Additional Mathematics curriculum in the near future. The objective of introducing Partial Fractions into the curriculum is to strengthen students' algebraic manipulation skills.

In terms of computation, this topic should not pose much difficulties to teachers as the skills involved are fairly similar to the chapter on Polynomials, Remainder and Factor Theorem. Readers may refer to any A Level Pure Math textbook.

In this note, I would like to discuss some issues that teachers might be interested to know about partial fractions. Here the readers are not assumed to recall much undergraduate abstract algebra. The computation aspects to find unknown constants are not dealt with here. Teachers may refer to any A Level Pure Math textbook.

Partial Fractions as the Reverse Process of Adding Algebraic Fractions

Secondary school mathematics students are familiar with the process of addition and simplification of algebraic fractions. For example,

$$\frac{1}{x-3}+\frac{1}{x+1}=\frac{(x+1)}{(x-3)(x+1)}+\frac{(x-3)}{(x-3)(x+1)}=\frac{2x-2}{(x-3)(x+1)}$$

Suppose we are given the fraction $\frac{2x-2}{(x-3)(x+1)}$ and asked to express this as the sum of the two fractions $\frac{1}{x-3}+\frac{1}{x+1}$. This is the whole purpose of this chapter of Partial Fraction, i.e. to be able to *resolve* an algebraic fraction into its *sum of partial fractions*.

Consider the algebraic fraction $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$. It can be expressed as $\frac{2x-3}{(x-1)(x-2)}+\frac{1}{x-3}$, $\frac{2x-4}{(x-1)(x-3)}+\frac{1}{x-2}$ or $\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$, which are all considered as *sums of partial fractions* of the $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$. fraction However, the third form $\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$ is said to have fully resolved the fraction $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$ *completely* into sum of its partial fractions. In our discussion in this note, we shall consider this complete resolution of partial fractions.

Fractions with Non-repeating Linear Polynomials as Denominators

Suppose we are asked to express $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$, where none of the k_i's are equal, into its sum of partial fractions. Then *to every linear factor* in the denominator, there corresponds one factor of the form $\frac{A_i}{x-k_i}$, that is,

$$\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}\equiv\frac{A_1}{x-k_1}+\frac{A_2}{x-k_2}+\frac{A_3}{x-k_3}+\;......\;+\frac{A_n}{x-k_n}\quad\text{-------(1)}$$

for some $A_1, A_2,......, A_n$

<u>Question 1:</u> Given any algebraic fraction of the form given by $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$, is it ***always possible*** to find constants A_1, A_2, ...,A_n such that (1) holds?

Combining the algebraic fractions on the right hand side of the equation (1) and equating the numerators we have

$$P(x)\equiv A_1(x-k_2)(x-k_3)...(x-k_n)+A_2(x-k_1)(x-k_3)...(x-k_n)+.......+A_n(x-k_1)(x-k_2)......(x-k_{n-1})\quad\text{-------(2)}$$

Observe that the equivalence of the two polynomials in (2) can only be possible ***only if*** P(x) is a polynomial of degree n – 1 or lower; otherwise the equivalence of (2) is impossible to achieve.

Before we answer Question 1, let us make one observation:

<u>Observation 1:</u> In order to be able to express $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$ fully into its sum of partial fractions, it is important that P(x) is of degree at most n – 1, that is, the algebraic fraction is a *proper fraction*. Otherwise, one needs to resolve the fraction into mixed fractions before resolving into partial fractions.[1]

<u>Illustration:</u> In trying to resolve $\frac{x^2-3x+3}{(x-1)(x-2)}$ into sum of its partial fraction, we check that the given fraction is *improper* in that the degree of numerator is not less than the degree

---

(Footnotes)

1 Note that in the above we have only shown that if $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$ can be expressed into sums of partial fractions, then P(x) must be of degree n – 1 or lower; the converse has not been proved yet.

of the denominator (both are of degree 2). Hence the first step is to express the fraction into mixed fraction, that is, $\frac{x^2-3x+3}{(x-1)(x-2)} \equiv 1 + \frac{1}{(x-1)(x-2)}$ and then express $\frac{1}{(x-1)(x-2)}$ into its sum of partial fractions. Some other questions that arises due to the above main question are discussed below:

We shall next attempt to answer Question 1 below:

(a) Given the algebraic expression , $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$ where degree of P(x) is of at most n – 1, is it ***always possible*** to express this completely into its sum of partial fractions given by the RHS of (1) above, i.e. of the form $\frac{A_1}{x-k_1}+\frac{A_2}{x-k_2}+\frac{A_3}{x-k_3}+ \;......\; +\frac{A_n}{x-k_n}$? Or are there some fractions of the form $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$ (with degree of P(x) at most n – 1) which cannot be expressed uniquely into its partial fractions?

(b) If an algebraic fraction $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$ can be resolved fully into sum of partial fractions of the form given by $\frac{A_1}{x-k_1}+\frac{A_2}{x-k_2}+\frac{A_3}{x-k_3}+ \;......\; +\frac{A_n}{x-k_n}$, are the choices of A_1, A_2,.........,A_n unique ?

Consider a simple algebraic fraction with two linear factors in the denominator, i.e. of the form $\frac{ax+b}{(x-k_1)(x-k_2)}$, where $k_1 \neq k_2$

Let $\frac{ax+b}{(x-k_1)(x-k_2)} \equiv \frac{A}{x-k_1}+\frac{B}{x-k_2}$, where the values of A and B are to be found. Combining the RHS of the above fraction we obtain $\frac{A(x-k_2)+B(x-k_1)}{(x-k_1)(x-k_2)}$. Then by equating the numerators of both sides of the equation we have

$$ax+b \equiv A(x-k_2)+B(x-k_1) \text{---------------------------------}(3)$$

Equating the coefficients of x and the constant term on both sides of equation (3), we have $\begin{cases} A+B &= a \\ k_2A+k_1B &= -b \end{cases}$, or written in the matrix form as $\begin{pmatrix} 1 & 1 \\ k_2 & k_1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$. The determinant of the coefficient matrix $\begin{pmatrix} 1 & 1 \\ k_2 & k_1 \end{pmatrix}$ is $k_1 - k_2$, which is always nonzero as long as $k_1 \neq k_2$. Thus, the numerical values of A and B of (3) can ***always be found <u>uniquely</u>***,[2] showing that the algebraic fraction $\frac{ax+b}{(x-k_1)(x-k_2)}$ can always be expressed into sum of partial fractions and this resolution is unique (so long as the linear factors are non-repeated, i.e. $k_1 \neq k_2$).

Consider the algebraic fraction with three non-repeating linear factors in the denominator given by $\frac{ax^2+bx+c}{(x-k_1)(x-k_2)(x-k_3)}$. Let

$$\frac{ax^2+bx+c}{(x-k_1)(x-k_2)(x-k_3)} \equiv \frac{A}{x-k_1}+\frac{B}{x-k_2}+\frac{C}{x-k_3} \text{ .----------------------------}(4)$$

Combining the RHS of the equation (4) we have $\frac{A(x-k_2)(x-k_3)+B(x-k_1)(x-k_3)+C(x-k_1)(x-k_2)}{(x-k_1)(x-k_2)(x-k_3)}$ and then equating the numerator of this expression with the LHS of (4) we have

$$A(x-k_2)(x-k_3)+B(x-k_1)(x-k_3)+C(x-k_1)(x-k_2) \equiv ax^2+bx+c.$$

Comparing the coefficients of x^2, x and the constant term on both sides of the above equations and upon further simplification yields

$$\begin{cases} A+B+C &= a \\ (k_2+k_3)A+(k_1+k_3)B+(k_1+k_2)C &= -b \\ (k_2k_3)A+(k_1k_3)B+(k_1k_2)C &= c \end{cases} \text{---------------------------------}(5)$$

Consider the coefficient matrix of the above system of equation which is given by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ k_2+k_3 & k_1+k_3 & k_1+k_2 \\ k_2k_3 & k_1k_3 & k_1k_2 \end{pmatrix}$. It can be found that the determinant of this matrix[3] has the value $(k_1-k_2)(k_1-k_3)(k_2-k_3)$. Thus, when none of the k_1, k_2 and k_3 are equal (that is, none of the linear factor in the algebraic fraction are repeating), this expression is nonzero. Hence there are unique solutions for A, B and C in the system (4) above. In other words, the algebraic fraction $\frac{ax^2+bx+c}{(x-k_1)(x-k_2)(x-k_3)}$ can always be fully resolved into sum of partial fractions uniquely.

In the same way, it can be shown that $\frac{P(x)}{(x-k_1)(x-k_2)(x-k_3)......(x-k_n)}$ (where degree of P(x) is at most n – 1) can always be fully resolved into its sum of partial fractions uniquely.

Fractions with Repeating Linear Polynomials as Denominators

To every linear factor that occurs n times in the denominator of an algebraic fraction (i.e. algebraic factor of the form $(ax+b)^n$), there corresponds a series of partial fractions $\frac{K_1}{ax+b}+\frac{K_2}{(ax+b)^2}+\frac{K_3}{(ax+b)^3}+ \;......\; +\frac{K_n}{(ax+b)^n}$.

For examples, given the algebraic fraction $\frac{2x+3}{(x-1)(x+2)^2}$, its corresponding partial fractions should be $\frac{A}{x-1}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$, since the factor $(x+2)^2$ occurs in the denominator; given the algebraic fraction $\frac{2x+3}{(x+2)^2(x-1)^3}$, where x + 2 occurs twice while $2x$ – 1 occurs three times in the denominator, its corresponding partial fractions should be given by $\frac{2x+3}{(x+2)^2(x-1)^3} \equiv \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$. We shall illustrate how a repeated factor turns out with so many partial fractions.

Consider $\frac{2x+3}{(x-1)(x+2)}$ where all the denominators consist of linear factors. Then its partial fraction will be

$$\frac{2x+3}{(x-1)(x+2)} \equiv \frac{A}{x-1}+\frac{B}{x+2} \text{ .-------------------------}(6)$$

(Footnotes)

[2] *It is a well-known fact that the system of simultaneous n linear equations in n unknowns expressed in the matrix form AX = B has a unique solution for X if and only if determinant of the coefficient matrix A is nonzero. For the case when n = 2, the system of equation* $\begin{cases} ax+by &= e \\ cx+dy &= -f \end{cases}$ *has unique solution if and only if the determinant of the coefficient matrix ad – bc is nonzero.*

[3] *In the O Level syllabus, the determinant of a 2x2 matrix can be computed fairly easily. In fact, determinants of any square n x n matrix can be computed, though the definition and the formular are much more complicated. O Level students are not required to know the formula for this computation. Here readers may take for granted that determinants for any square matrix can be computed.*

In the previous section we saw that the choices of A and B can be determined uniquely. Then multiplying by $\frac{1}{x+2}$ to both sides of (6), we obtain

$$\frac{2x+3}{(x-1)(x+2)^2} \equiv \frac{A}{(x-1)(x+2)} + \frac{B}{(x+2)^2} \text{ ------------------(7)}$$

And repeating the expressing of the fraction $\frac{A}{(x-1)(x+2)}$ in (7) above further into partial fractions we have

$$\frac{2x+3}{(x-1)(x+2)^2} \equiv \frac{A_1}{(x-1)} + \frac{A_2}{(x+2)} + \frac{B}{(x+2)^2} \text{ , --------------(8)}$$

which accounts for the "extra" partial fractions corresponding to the repeated linear factor.

Consider $\frac{2x+3}{(x-1)(x+2)^2}$. Suppose we start from (8), we have $\frac{2x+3}{(x-1)(x+2)^2} \equiv \frac{A_1}{(x-1)} + \frac{A_2}{(x+2)} + \frac{B}{(x+2)^2}$. Multiplying both sides by $\frac{1}{x+2}$ and breaking down the second algebraic fraction further into partial fractions, we obtain

$$\frac{2x+3}{(x-1)(x+2)^3} \equiv \frac{A_1}{(x-1)(x+2)} + \frac{A_2}{(x+2)^2} + \frac{B}{(x+2)^3} \text{ .---------(9)}$$

Notice that the fraction $\frac{A_1}{(x+2)(x-1)}$ in (9) and can further be broken down into partial fractions $\frac{B_1}{x-1} + \frac{B_2}{x+2}$, so that (9) becomes

$$\frac{2x+3}{(x-1)(x+2)^3} \equiv \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} \text{ --------------------- (10)}$$

after suitable renaming of the constant terms in (9).

Inductively, it becomes clear that if there is a linear factor in the denominator that occurs n times, for example $\frac{2x+3}{(x-1)(x+2)^n}$, consider then its resolution completely into partial fractions is of the form

$$\frac{2x+3}{(x-1)(x+2)^n} \equiv \frac{A}{(x-1)} + \frac{B_1}{(x+2)} + \frac{B_2}{(x+2)^2} + \frac{B_3}{(x+2)^3} + \ldots\ldots + \frac{B_n}{(x+2)^n}$$

Thus, the "strange" phenomenon of having a series of "extra" partial fractions for the case of a repeated linear factor in the denominator can be explained inductively in the above fashion. Of course, teachers might want to choose a simpler example as an illustration for students.

There are other cases for partial fractions in which the denominators are of irreducible quadratic polynomials. These cases are not discussed in this issue.

Related Issue involved in Partial Fractions

There are issues in teaching this chapter which might not be directly related to partial fractions.

The process of expressing an improper algebraic fraction into mixed fraction might need to be reinforced in teaching Partial Fractions. Firstly as discuss above, it is important to check a given fraction as proper before expressing an algebraic fraction into its sum of partial fractions.

Another possible problem is that some students might mistake quadratic polynomials that are not factorizable over the set of rational numbers (characterized by those quadratic polynomials that can be factorized by the trail-and-error method) as *irreducible* quadratic polynomials instead of using completing squares to factorize the polynomial. For example,

$$x^2 - 2x - 5 = (x - 1 - \sqrt{6})(x - 1 + \sqrt{6}).$$

Perhaps teachers might want to familiarize their students with factorization of polynomials that are not factorizable over the rational numbers but are only factorizable over the real numbers.

Conclusion

In this note, we have discussed some issues related to Partial Fractions. The readers might want to refer to any A Level Pure Mathematics textbooks for more information about this chapter.

Integrating Technology into the Mathematics Classroom

Lyncoln Lee, Catholic Junior College

Dr Stephen demonstrating the use of the Ti84 Plus at CJC

In a workshop jointly organized by the Association of Mathematics Educators (AME) and Catholic Junior College, Dr Stephen Arnold shared his experience in using technology in the Mathematics classroom with about 20 Mathematics teachers from various Junior Colleges on 7th December 2005. The workshop focused mainly on the use of the TI-84 Plus Graphic Calculator from Texas Instruments. It aims to provide teachers with an insight on the capabilities of the calculator as well as some examples on how to use the calculator to teach Mathematical concepts, as the Graphic Calculator will be an essential component in the new A-Level syllabus in 2006.

Dr Stephen demonstrated the diverse use of the Graphic Calculator in the teaching of calculus, probability, statistics, and geometry using real-world problems. He also introduced the use of sensors and probes to collect data using the TI84 Plus for the teaching of Mathematics.

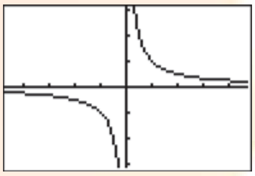
Algebra and Graphing

For students who are new to graphs, the teacher can get students to plot the following curves using the calculator. In order of increasing difficulty:

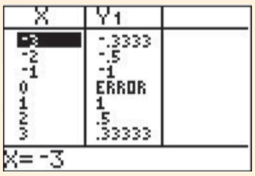
i. $y=x$
ii. $y=2x-3$
iii. $y=4-3x$
iv. $y=x^2-3$

A discussion can then follow, to bring out ideas like gradients, turning points and translation by comparing the different curves with the basic $y=x$ curve.

For reciprocal functions or "inversions", the concept of asymptotes can be introduced. By graphing the basic $y=\frac{1}{x}$ curve, it may not be obvious to students that $x=0$ is an asymptote.

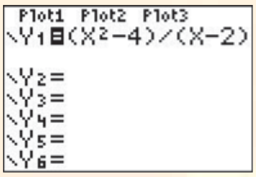


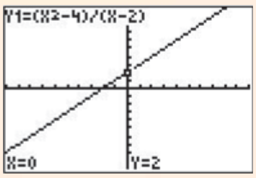
However, by switching to the table of values, students can see that the function is undefined at $x=0$.

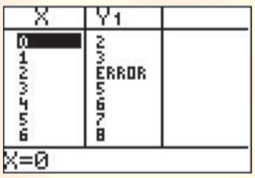


Calculus – The Sky is the Limit

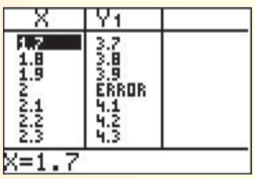
Begin with a function which has a "hole", such as $y=\frac{(x^2-4)}{(x-2)}$ At first glance, it *looks* like the linear function $y=x+2$, until the table of values is consulted.



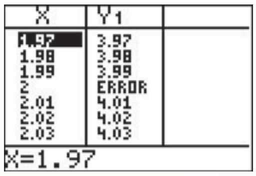




So what exactly is happening at the point $x=2$? Use the table of values to "zoom in" for a closer look.



Further zooming in:



The students should be able to observe that both above and below the critical value, the function moves towards a fixed value of 4. It is important for the students to realise that while the function approaches the value of 4 but never reaches it, the value of 4 *is* the limit of the function at $x=2$. A common misconception for students is that the limiting value is an approximation, rather than an exact real value.

Probability – The Birthday Problem

What is the probability that, in a group of people, two will share the same birthday?

One approach is to consider the reverse: What is the chance of two people NOT sharing the same birthday?

Begin by imagining a single person in a room. Another enters the room. What is the likelihood that they have different birthdays? The number of days the first person may choose for a birthday is unrestricted: 365. Then the second person may choose from only 364 days. The probability is therefore

$$\frac{365}{365} \cdot \frac{364}{365}$$

Then the probability of them sharing the same birthday will be the compliment of this result.

$$1 - \frac{365}{365} \cdot \frac{364}{365}$$

Another person joins them. The chance of them having different birthdays is now

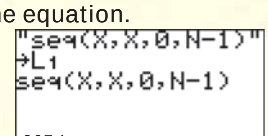
$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}$$

So in general, for a group of n people, the likelihood of two people sharing the same birthday is

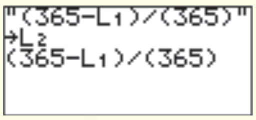
$$1 - \prod_{k=0}^{n-1} \left(\frac{365-k}{365} \right)$$

It is possible to create a small program to calculate the probability using the equation above. Start by assigning a value to n, the number of people in the group. For example, let $n=10$.

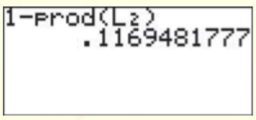
Input the following statement to create a sequence of numbers from 0 to n-1 and then store it in List 1. List 1 will now contain all values of k in the equation.



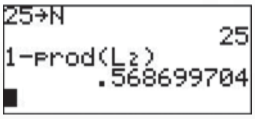
Store the values of $\frac{365-k}{365}$ in List 2 by inputting the following statements.



The required probability can be obtained easily by subtracting the product of all values in List 2 from 1.

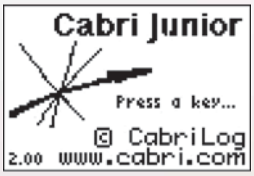


By substituting different values of n, we can obtain the required probability very quickly for different groups of people. This can be a very good tool for exploring the problem. It may come as a surprise, but for a group of 30 people, there is actually a 70% chance of finding a pair of birthday buddies!

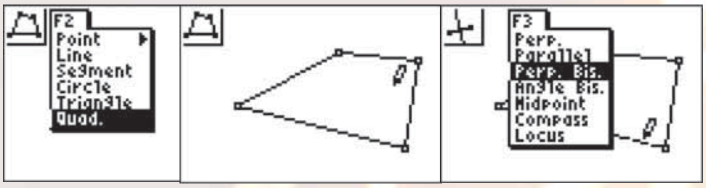


Interactive Geometry

Geometry can also be taught using the Graphic Calculator by accessing Cabri Junior in the applications menu.



Cabri Junior will allow students to explore and create various geometrical shapes. It can also perform many geometrical construction and transformation operations. Teachers can also create animations in Cabri Junior to illustrate concepts like loci.



Trigonometric Functions - Harmony All Around

Finally, Dr Stephen demonstrated the use of different sensors and probes to collect real-world data by connecting a data logger to the Ti-84 Plus. These included motion detectors, light sensors and microphones.

Dr Stephen using a motion detector to record the motion of a slinky

Motion detectors can be used to record simple harmonic motions around us. The motion of a slinky or a swinging handbag makes a very good introduction to trigonometric functions.

Light Sensors placed under a fluorescent light produced a sinusoidal wave since fluorescent lights flicker at a standard rate of 50 cycles per second (50 Hz). Microphones used to record a steady note will also produce a sinusoidal wave. Get different students to sample their voices using the microphone and get them to compare the difference.

Teaching Mathematics using such an approach blurs the lines between mathematics and science, but the ways in which mathematics models the real world is too important to forgo. Mathematics should not be taught in isolation, without relating to the real world. Teaching mathematics in an algorithmic way is a “rejection of meaning” to what students are learning; and it devalues computational skills of students and provides no opportunities for them to transfer their knowledge.

Dr Stephen believes that the use of technology is an essential part of teaching Mathematics. Although many teachers are able to teach mathematics effectively without the use of technology, it is too powerful a tool to pass up. It is a catalyst for change, and provides students with the opportunity for working with real world problems. The employment of technology can make Mathematics more experimental, so that students can build understanding through trial and error. They can also learn to apply their knowledge through the real life experiences. Ultimately, he hopes that students will be able to change their perspective of the world around them with their knowledge of mathematics, and grow to gain a richer appreciation of the world we live in.

“Golden Rules” when using Technology

- Be convinced that technology is important.
- Start with the Mathematics, NOT the button pushing.
- Always encourage student participation.
- Before pressing the buttons, always ask students what they expect to see.
- After obtaining the results, ask students if it coincides with what they expected. Why? Why not?
- Get students to work in pairs or small groups and encourage verbalisation of thoughts.

Dr Stephen Arnold has been a teacher for 27 years, and has been actively using classroom technology for almost 20 years. In addition to teaching mathematics, science and technology in a range of Australian high schools, Dr Stephen has been involved in teacher education at several Universities. He is currently Head of the School of Education at the Australian Catholic University in Canberra. He has published and presented widely across Australia, New Zealand and South-East Asia.

The answer is:
8 years old, 3 years old and 3 years old

Hints:
1. the product of their ages is 72
2. The census still does not have enough information to solve the problem when the mathematician told him that the sum of their ages is the same as the house no
3. the oldest child likes ice-cream

Mathematics Teachers Conference 2006

Theme: Enhancing Mathematical Reasoning

1st June 2006
National Institute of Education
1 Nanyang Walk
Singapore

Jointly organized by:

- Association of Mathematics Educators
- Mathematics and Mathematics Academic Group,National Institute of Education, NTU

Programme

0800 – 0845 h Registration

0900 – 0930 h Opening of Conference

0930 – 1000 h Tea Break

1000 – 1100 h Keynote I

Primary -
The nature of mathematical reasoning & how to teach it
Dr Yeap Ban Har & Mr Lee Ngan Hoe

Secondary -
Enhancing mathematical reasoning at secondary school level
A/P Wong Khoon Yoong

Junior College -
Mathematical reasoning in pre-university education
by Prof Ling San

1100 – 1200 h Keynote II

Primary -
The nature of mathematical reasoning & how to teach it (contd)
Dr Yeap Ban Har & Mr Lee Ngan Hoe

Secondary -
Enhancing students' reasoning via thoughtful problem-solving activities
A/P Berinderjeet Kaur

Junior College -
Mathematical reasoning: From secondary to pre-university mathematics education
Dr Toh Tin Lam

1200 – 1400 h Lunch & Tour of Stalls (Books & Teaching Materials)

1400 – 1600 h Concurrent Workshops - Participants to attend one

Primary

P1 - **Developing mathematical reasoning – The role of calculators**
by A/P Ng Swee Fong

P2 - **Using short open-ended questions to develop and assess mathematical reasoning**
by A/P Foong Pui Yee

P3 - **Enhancing mathematical reasoning through journal writing in the primary math classroom**
by A/P Douglas Edge & Mr Eric Chan

P4 - **Games in primary mathematics classrooms**
by A/P Koay Phong Lee

P5 - **Generating mathematics investigative tasks from a given stem**
by Ms Chua Kwee Gek

Secondary

S1 - **Geometric reasoning through folding circles**
by Ms Teo Soh Wah & Mrs Tan Kum Fong

S2 - **Mathematical reasoning in algebra**
by Dr Toh Tin Lam

S3 - **Using investigative tasks to enhance mathematical reasoning**
by Mr Joseph Yeo B.W.

S4 - **Strategies to enhance students' reasoning in mathematics at the secondary level**
by Dr Jaguthsing Dindyal

S5 - **Engaging lower secondary pupils through mathematical reasoning**
by Dr Joseph Yeo Kai Kow

S6 - **Use of IT in the teaching of statistics**
by A/P Yap Sook Fwe

Junior College

J1 - **Using a graphing calculator to extend mathematical reasoning skills**
by Dr Ng Wee Leng

J2 - **Newton's binomial series and the computation of pi**
by Dr Paul Shutler

1600 – 1630 h Tea break

1630 – 1700 h Closing address & presentation of tokens of appreciation.

Abstracts of keynote lectures and workshops are available at our webpage: http://math.nie.edu.sg/ame/
You may also download registration form and details.
You are advised to register early as places are limited.
Closing date for conference registration: **Saturday 15th April 2006.**

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