ASSOCIATION OF MATHEMATICS EDUCATORS
Website: http://math.nie.edu.sg/ame/

Maths Buzz

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Contents

President's Message...

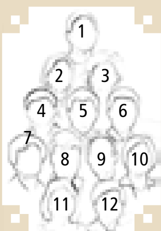
Dear AME Members,
The Association of Mathematics Educators is now 12 years old. Under the able leadership of the past 6 executive committees the Association has gained not only national but also international recognition. Our publications, Maths Buzz and The Mathematics Educator have both served their purposes well. The Maths Buzz has helped to wet the appetite of our members – kept them posted briefly about upcoming events and shared with them some interesting snippets on mathematical ideas or teaching episodes. The Mathematics Educator has been very successful in publishing research papers related to the teaching and learning of mathematics. It is currently a much sought after publication by international researchers. This is certainly a significant milestone for the Association. To all our past and present editors of Maths Buzz and The Mathematics Educator we say, **Thank You and Well Done**!

In line with its aims and goals, the Association has to date provided numerous professional development and enrichment activities for members and mathematics teachers. It has organized numerous workshops by local as well as international experts. It has also co-organized both national and international conferences such as the ERA-AME-AMIC Conference in 1999 with Educational Research Association of Singapore and ICMI - EARCOME in 2002 with Mathematics & Mathematics Education AG at NIE. The contributions of the 6th executive committee in making an annual Mathematics Teachers Conference as part of the activities of the Association is laudable and deserve commendation! The first such conference in Singapore was held on 2nd June 2005. The theme of the conference was *Assessment* which signaled a need for teachers to update and prepare themselves to engage in Alternative Modes of Assessment as outlined in the 2004 Mathematics Assessment Guides issued by the Ministry of Education. The same committee organized the second conference held on 1st June 2006 the theme of which was *Enhancing Mathematical Reasoning*. Both conferences were very well attended and received by members of the Association and mathematics teachers.

At the 13th annual general meeting of the Association, I was elected President of the seventh executive committee of the Association. This is my third time as President of the Association. The seventh executive committee of the Association will strive to do their best for the Association. We are already planning the next Mathematics Teachers Conference to be held on 31st May 2007 with the theme: *Mathematical Literacy*. As we work towards the aims and goals of the Association we need your support and cooperation for the continued well-being of the Association and to scale it to greater heights. We value your views and feedback. Please feel free to e-mail us about any matter related to the Association. The Association's homepage is http://math.nie.edu.sg/ame/ .

A/P Berinderjeet Kaur
President
Association of Mathematics Educators (2006 – 2008)
Nanyang Technological University

Executive Committee 2006-2008



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Three Panels Presentation

Mr Leong Yew Hoong
National Institute of Education

Presenting Mathematics using different 'modes'

As teachers, we try to present mathematical ideas in ways that students find easier to make sense of. Sometimes we use alternative 'modes' of instruction to present the subject matter in ways that are easier to access by the students. I use the term 'mode' here in a very restrictive sense to mean the different visual forms that we as teachers use to communicate mathematical content. There are many such modes found in the literature and also different terms used to label these modes. In my discussion here, I refer to the following modes:

I. "Textual" – written texts that are summaries or salient points of our verbal instruction;
II. "Diagrammatic" – diagrams, pictures, graphs;
III. "Numerical" – working steps or examples involving only numbers
IV. "Symbolic" – working steps or examples involving abstract symbols

There are admittedly other modes of representation commonly used in the classroom such as the technology mode and the enactive mode, among others. For the purpose of this article, however, I will narrow the scope of discussion only to modes that are easily presented on the board. Also, not all the modes I-IV above is relevant in any particular teaching episode. Depending on the subject matter, some modes are more suitable than others. The examples below may better illustrate the modes and their respective uses.

An example of the use of different modes: adding algebraic fractions

In introducing the method of adding algebraic fractions, some teachers find the immediate use of algebraic fractions as a first example rather daunting for students. So they prefer to start with a simpler numerical example to demonstrate the addition of numeric fractions. They then tease out the underlying steps involved in the process before applying the same process to the addition of algebraic fractions. This sequence is in line with the familiar-to-unfamiliar and concrete-to-abstract progression of instruction as encouraged by many as a sound pedagogical principle. In a more diagrammatic form, the progression can be illustrated as follows:

Start with adding numeric fractions	→	Tease out the underlying process	→	Apply the process to addition of algebraic fractions

Interpreted through the language of instructional modes I discussed earlier, the teaching sequence can also be seen as a movement from the numerical mode, to textual mode, and then the symbolic mode:

Numerical	→	Textual	→	Symbolic

The possible visual forms illustrated under each mode are given below as a reference to the reader.

Numerical:

$$\frac{1}{2}+\frac{3}{5}$$

[LCM(2,5) = 10]

$$=\frac{1\text{x}5}{2\text{x}5}+\frac{3\text{x}2}{5\text{x}2}$$

$$=\frac{5}{10}+\frac{6}{10}$$

$$=\frac{5+6}{10}$$

$$=\frac{11}{10}$$

Textual:

Step 1 obtain the LCM of the denominators
Step 2: convert each fraction to equivalent fractions with same denominators
Step 3: form one fraction by adding numerators

Symbolic:

$$\frac{x}{2}+\frac{2x}{3}$$

[LCM(2,3) = 6]

$$=\frac{x\text{x}3}{2\text{x}3}+\frac{2x\text{x}2}{3\text{x}2}$$

$$=\frac{3x}{6}+\frac{4x}{6}$$

$$=\frac{3x+4x}{6}$$

$$=\frac{7x}{6}$$

Linking the different modes more tightly: '3 panels presentation'

While a teacher who uses the above progression intends to show a deliberate coherence in the instructional sequence, the problem is that students sometimes do not see the implicit connections clear enough. When the teacher is presenting a particular mode of representation, students may have forgotten the previous mode(s) discussed. In other words, perhaps teachers may need to show the links between the modes more explicitly in their teaching. The danger of simply showing the different modes one after another in class is that students will then view the modes in isolation and not as tightly-linked as is intended by the teacher.

I propose a '3 panels presentation' on a white/blackboard that can perhaps strengthen the links between modes. Since the intention here is to demonstrate in sequential order what should appear on the whiteboard, the following illustration may be better than description by words:

1st panel:

"Numerical" mode

Numeric fractions		
$\frac{1}{2}+\frac{3}{5}$ [LCM(2,5) = 10] $=\frac{1\text{x}5}{2\text{x}5}+\frac{3\text{x}2}{5\text{x}2}$ $=\frac{5}{10}+\frac{6}{10}$ $=\frac{5+6}{10}$ $=\frac{11}{10}$		

2nd panel:

"Numerical" mode	"Textual" mode	
Numeric fractions	**Commentary**	
$\frac{1}{2}+\frac{3}{5}$		
[LCM(2,5) = 10] ←	Step 1: obtain the LCM of the denominators	
$=\frac{1\text{x}5}{2\text{x}5}+\frac{3\text{x}2}{5\text{x}2}$ ←	Step 2: convert each fraction to equivalent fractions with same denominators	
$=\frac{5}{10}+\frac{6}{10}$		
$=\frac{5+6}{10}$ ←	Step 3: form one fraction by adding numerators	
$=\frac{11}{10}$		

3rd panel:

"Numerical" mode	"Textual" mode	"Symbolic" mode
Numeric fractions	**Commentary**	**Algebraic fractions**
$\frac{1}{2}+\frac{3}{5}$		$\frac{x}{2}+\frac{2x}{3}$
[LCM(2,5) = 10] ←	Step 1: obtain the LCM of the denominators	→ [LCM(2,3) = 6]
$=\frac{1\text{x}5}{2\text{x}5}+\frac{3\text{x}2}{5\text{x}2}$ ←	Step 2: convert each fraction to equivalent fractions with same denominators	→ $=\frac{x\text{x}3}{2\text{x}3}+\frac{2x\text{x}2}{3\text{x}2}$
$=\frac{5}{10}+\frac{6}{10}$		$=\frac{3x}{6}+\frac{4x}{6}$
$=\frac{5+6}{10}$ ←	Step 3: form one fraction by adding numerators	→ $=\frac{3x+4x}{6}$
$=\frac{11}{10}$		$=\frac{7x}{6}$

By presenting all the 3 modes in 3 corresponding panels on the board, it is easier for teachers to point out the explicit inter-modal links and the overall 'story' in the progression between panels. Students can also make the links and view the overall progression easier by making visual references to the connections illustrated on the board.

Another example: solving a 'word problem'

Suppose we want to teach students (who are just beginning to grapple with algebra) the way to transform the simple word problem shown below into an algebraic equation.

"Peter is 7 years older than John. Given that the sum of their ages is 33, find their respective ages."

Instead of doing a one-step transformation from the above word problem into an algebraic equation, the teacher might want to break down the problem into more easily analysable parts as well as mediate the 'jump' between textual and symbolic modes by the more-familiar diagram mode of the 'models'. The three panels presentation may possibly be useful for the purpose of providing the gradual progression, with the final appearance of the board as shown below.

"Textual" mode	"Diagrammatic" mode	"Symbolic" mode
Phrases	**Model**	**Algebra**
John's age →	$\boxed{x}$	→ x
Peter is 7 years older. →	$\boxed{x \;\vert\; 7}$ } 33	→ $x + 7$
Sum of their ages is 33 →		→ $x + (x + 7) = 33$

Although the "x"s in the rectangles are shown in the above under the second panel, the more appropriate timing of their actual insertion should be when the algebraic process is carried out under the third panel.

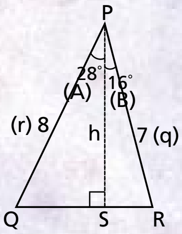
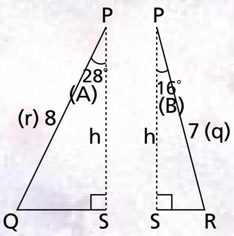
Since the two examples of teaching above dealt with topics in the lower secondary syllabus, it is perhaps fitting to end the article with an example set within the upper secondary context.

Another example: teach a proof of
"sin(A+B) = cos$B$ sin$A$ + cos$A$ sin$B$"

Teachers who attempt to teach proofs often find it frustrating that some students do not seem to be able to 'get it'. Although the inter-step progression in the formal process appears simple, yet it is not always easy for students to overcome the sea of algebraic symbols as well as to appreciate the beauty of the overall 'attack strategy'. In the three panels presentation shown at the appendix, there is an attempt to provide the overall proof strategy by showing the diagrammatic overview. The second panel gives some kind of a numerical 'dry run' of the written steps so as to cushion the sudden introduction of algebraic symbols. The actual formal proof process is only presented in the third panel as a natural progression of the previous modes. Obviously, before the teacher starts to write on the third panel, "A" and "B" should be appropriately inserted (perhaps with a different colour marker) beside the "28°" and the "16°" respectively to signal the movement from a numerical case to the generalized context. The length of the sides measuring "7 cm" and "8 cm" will also be correspondingly matched to "q" and "r" respectively.

The original ideas in this article came about through my teaching interactions with student teachers at NIE over a few PGDE cohorts. Their concerns with inter-modal links prompted a collective inquiry into practical ways to alleviate the problem. An earlier form of this article was presented at the East Zone Mathematics Centre of Excellence sharing day held at Tao Nan School on 2 Jun 2006. I thank the teachers who attended my session and provided helpful comments, most of which are incorporated in this revised article.

Appendix

"Diagrammatic" mode	"Numerical" mode	"Symbolic" mode
Overview	**Numerical example**	**Proof**
	Area of ΔPQR = Area of ΔPSQ + Area of ΔPSR $\frac{1}{2} \times 8 \times 7 \times \sin(28^\circ + 16^\circ) = \frac{1}{2} \times 8 \times h \times \sin 28^\circ + \frac{1}{2} \times 7 \times h \times \sin 16^\circ$ $\div \frac{1}{2}$: $8 \times 7 \times \sin(28^\circ + 16^\circ) = 8 \times h \times \sin 28^\circ + 7 \times h \times \sin 16^\circ$	Area of ΔPQR = Area of ΔPSQ + Area of ΔPSR $\frac{1}{2} rq \sin(A + B) = \frac{1}{2} rh \sin A + \frac{1}{2} qh \sin B$ $\div \frac{1}{2}$: $rq \sin(A + B) = rh \sin A + qh \sin B$
	[In ΔPSR, $\cos 16^\circ = \frac{h}{7}$, so $h = 7 \times \cos 16^\circ$ In ΔPSQ, $\cos 28^\circ = \frac{h}{8}$, so $h = 8 \times \cos 28^\circ$] So, $8 \times 7 \times \sin(28^\circ + 16^\circ) = 8 \times 7 \times \cos 16^\circ \times \sin 28^\circ + \times 7 \times 8 \times \cos 28^\circ \times \sin 16^\circ \div (8 \times 7)$: $\sin(28^\circ + 16^\circ) = \cos 16^\circ \times \sin 28^\circ + \cos 28^\circ \times \sin 16^\circ$	[In DPSR, $\cos B = \frac{h}{q}$, so $h = q \cos B$; and In DPSQ, $\cos A = \frac{h}{r}$, so $h = r \cos A$] So, $rq \sin(A + B) = rq \cos B \sin A + qr \cos A \sin B \div rq$: $\sin(A + B) = \cos B \sin A + \cos A \sin B$

Connecting Mathematics and Science

Quek Wei Ching
Singapore Polytechnic

On March 20, the first Monday of the term, the Association of Mathematics Educators and the Department of Mathematics & Science, Singapore Polytechnic have jointly presented a Graphing Calculator workshop "Connecting Mathematics and Science" by Mr Russell Brown from Bendigo Senior Secondary College, Victoria, Australia. About 40 educators took a break from their busy schedules to attend this workshop.

The workshop covered general secondary level mathematics using the Graphics Calculator and incorporated electronic data collection and analysis as an ideal way to enhance the kinaesthetic and visual aspects of teaching and learning. With a data logger, participants had a fun afternoon adjusting the speed and directions of their motions to match a velocity-time graph, to model the motion of bouncing ball, to measure the frequency of light source in the classroom. Some data were collected at time intervals of three thousandths of a second – the whole collection time was three hundredths of a second!

There was interesting sharing on choosing suitable axes for graphing, and exploration of a simulation program on probability.

During the workshop, we gained lots of exposure to the capabilities and resources available to incorporate graphing calculators to deliver meaningful mathematics and science lessons.

Interesting Interpretation of the Binomial Expansion of $(x+y)^n$

Yap Hui Hui
Innova Junior College

When I was a student in JC, I have always found the topic Binomial Theorem to be very dry and involves a lot of routine working and could not find any meaning in it. However, inspired by the combinatorics course in my university days, I discovered an interesting way to interpret the Binomial expansion of $(x+y)^n$, where n, x, y are positive integers .

> Let n, x, and y be positive integers. Then
> $$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n-1}xy^{n-1} + y^n$$

Let us try to find the expansion for $(x+y)^3$ first. We can attempt to solve this problem by transforming this into the following real life situation.

Suppose in a class of x girls and y boys, there are 3 class positions to be filled up, namely, monitor, assistant monitor and treasurer. How many ways can the teacher choose the students in the class to take up the positions, assuming that a student can take more than 1 position?

Solution:
There are 3 positions to be taken up. For each position, there are $(x+y)$ choices, since we assume each student can take up more than 1 position. Hence, by Multiplication Principle, total number of ways in which the 3 positions are taken up

$$= (x+y)^3 \text{ -------------- (1)}$$

However, we can solve this problem from another angle by breaking up into 4 cases, namely, for the 3 positions, there is no boy, 1 boy, 2 boys or 3 boys.

Case 1: **No** position is taken up by boys, i.e. all are taken up by girls.
Number of ways = x^3.

Case 2: **Exactly 1** position is filled up by boy.
Number of ways = $\binom{3}{1}x^2y$.

Case 3: **Exactly 2** positions are filled up by boys.
Number of ways = $\binom{3}{2}xy^2$.

Case 4: **All 3 positions** are filled up by boys.
Number of ways = y^3.

By Addition Principle, total number of ways in which the 3 positions are taken up

$$= x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + y^3 \text{ -------- (2)}$$

But (1) = (2), that is,

$$(x+y)^3 = x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + y^3 .$$

Similarly, in general, for the expansion of $(x+y)^n$ for positive integers n, x and y, we consider the situation, where in a class of x girls and y boys, there are n class positions to be filled up. How many ways can the teacher choose the students in the class to take up the positions, **assuming that a student can take more than 1 position**?

Using the same method as above, this means that

1. by multiplication principle, there are $(x+y)^n$ ways to do this.
2. there will be $(n+1)$ terms in the expansion, since there are $(n+1)$ cases to be considered, i.e. for the n class positions, there could be 0 boy, 1 boy, . . ., or n boys.
3. we can obtain the general term, T_{r+1}, $r = 0, 1, \ldots, n$, of the expansion by considering the general case where among the n positions, exactly r positions are taken up by boys. That is, $T_{r+1} = \binom{n}{r}x^{n-r}y^r$.
4. By Addition Principle and point 1,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots + \binom{n}{n-1}xy^{n-1} + y^n$$

The above way of deriving the identity is in fact quite a common practice in combinatorics, known as "counting it twice".

This method of deriving the identity is only limited to the case when x, y and n are positive integers. Interested readers are therefore invited to build upon and extend to a more general case.

Squeeze your Brain

Who's married and to whom?

Alan, Brandon, Calvin, Daniel, Elvin and Fred are married to Annie, Betty, Cathy, Denise, Eva and Florence, though not necessarily in that order. Denise, Daniel and Fred are triplets and their youngest uncle is Brandon. Florence is a first cousin of these triplets. Florence's sister is married to Elvin. Annie is a first cousin of Alan.

Denise has three children. Daniel has two primary-school going children. Elvin already has four young children with another due in 3 months' time. Annie is enjoying her role as a mother but cannot imagine having more than two children. Fred and his wife have chosen not to have any children.

Betty grew up with many siblings and like her mother and husband, she strongly believe that children are God's gifts. Thus Betty and husband already have several young children and they often joke that their children can form a basketball team. Eva has two teenage children who are studying in junior colleges.

Alan is just newly married and does not plan to have any children in the next 2 years. Alan has never introduced his wife to Eva, who is carrying on an extramarital affair with Elvin. Annie is considering telling Elvin's wife about it.

Answer the following questions:
(1) Who is married to whom?
(2) Who's the unfortunate husband of Eva?
(3) Who is Florence's sister?

Adapted by Lim Heng Leng (PGDE '06-'07) from the book Effective Problem Solving by M. Levine.

Flaws in Algebra Assessment Items

Chua Boon Liang
National Institute of Education

"Defective assessment items in mathematics" by Dr Jaguthsing Dindyal in the May 2006 issue of this publication is a meaningful and useful article in the wake of the 2005 Primary School Leaving Examination (PSLE) incident, where a flawed problem involving a rectangle with incorrect dimensions crept into the mathematics paper. With a detailed discussion of the inherent flaws in four geometry test items, the article also seems a timely reminder to mathematics teachers about the need to exercise greater caution when constructing test items to ensure validity. An examination of some current secondary school mathematics test papers, set between January and May 2006, shows encouraging evidence that the vast majority of the test items were well-designed and correct. Having said this, there is also reason to be somewhat concerned about the erroneous items in the test papers because many of these items seem apparent in two specific strands of mathematics: geometry and algebra. Since the typical flaws related to geometry test items had already been dealt with in Dr Dindyal's article, this present article will therefore focus on those flaws commonly seen in algebra test items. Ten items, whose flaws are classified into four categories, are discussed.

Case 1: Inappropriate instructions

Item 1

Find the product of:

(a) $-3x(4y)$ [1]

(b) $(-4xy)(-xy^2)$ [1]

Item 2

Rationalise the following expression: $\dfrac{3}{\sqrt{2}-\sqrt{5}}$. (3)

Item 3

Express each of the following as a single fraction in its lowest terms.

(a) $\dfrac{25x}{10xy} =$ [2M]	(b) $\dfrac{169m^3n}{13mn^2} =$ [2M]
(c) $\dfrac{4}{4k+4p} =$ [2M]	(d) $\dfrac{9s+3t}{15t+9s} =$ [2M]

In algebra, the product of x and y refers to the result when x is multiplied by y, which is denoted by xy. Thus the phrasing in Item 1 might confuse and mislead pupils to think of multiplying $-3x(4y)$ by $(-4xy)(-xy^2)$ when actually they are supposed to simplify the two expressions separately. Therefore, to give pupils clearer instruction on what to do in this test item, the correct direction is to ask them to simplify the expressions.

The phrasing of the problems in the next two test items might also be misleading. The purport of the instruction in Item 2 seems to be this: express $\frac{3}{\sqrt{2}-\sqrt{5}}$ as a fraction of two integers. Yet, it is impossible to perform this task because the expression can never be transformed into such a fraction. However, the expression can be simplified to $-(\sqrt{2}+\sqrt{5})$ by rationalising its denominator, and this process is understood to be the item writer's intent. Therefore, the instruction in Item 2 should be revised to clearly indicate this meaning. Similarly, it does not quite make sense to ask pupils to express the four expressions in Item 3 as single fractions when they are already single fractions themselves. Rather, when asking pupils to start with a single algebraic fraction such as $\frac{25x}{10xy}$ and write it in its simplest form $\frac{5}{2y}$, a proper instruction should be to ask pupils to simplify it.

Case 2: Misuse of mathematical terms

Item 4

Solve the following equations.

(a) $\dfrac{a^2+ab}{ab+b^2}$ [2] (b) $\dfrac{2c^2-50}{3(c+5)(c-10)}$ [2]

Item 5

Given the quadratic expression

$$3x^2+5x-c=0,$$

Solve x in terms of c by completing the squares and hence find the range of value(s) of c where the expression has two real roots. [4]

Item 6

Factorize the following quadratic equations.

(a) $x^2+5x-24$, [2]

(b) $6y^2-y-2$, [2]

(c) $9mn+12m+6n+8$ [2]

(d) $ab-2ad-3bc+6cd$ [2]

Item 7

The expression $f(x)=2x^3+ax^2-11x+6$ leaves a remainder of -42 when divided by $x+3$. Find the value of a. [3]

Item 8

Solve the following quadratic equation , giving your answers correct to 2 decimal places where applicable.

(a) $(3x+2)(6-x)=16$ [4]

(b) $\dfrac{1}{y}-\dfrac{2}{y+1}=3$ [3]

The incorrect use of the terms "expression" and "equation" in Items 4 to 6 reveals a common misunderstanding of the difference between an expression and an equation among some mathematics teachers. An algebraic expression is a mathematical statement involving numbers, letters or symbols, and operational signs. On the other hand, an algebraic equation is a mathematical statement expressing the equivalence of two expressions. So in light of these definitions, $\frac{a^2+ab}{ab+b^2}$ (in Item 4),

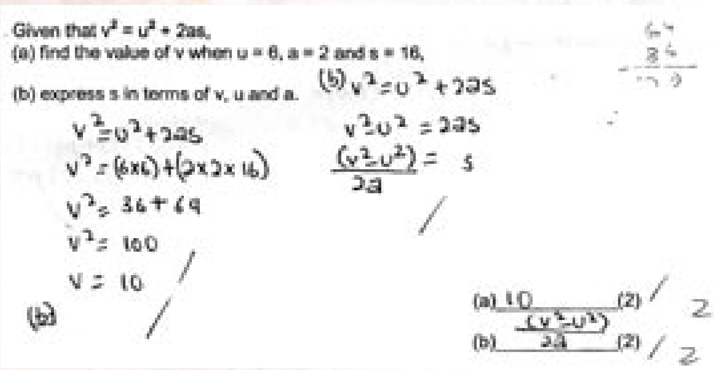
$x^2+5x-24$ and $9mn+12m+6n+8$ (both in Item 6), as well as the rest in the same two test items, should be termed as algebraic expressions whereas $3x^2+5x-c=0$ (in Item 5) is an equation rather than an expression. Now, it should also be clear why $f(x)=2x^3+ax^2-11x+6$ in Item 7 is not an expression. But neither is it appropriate to call that statement an equation because of the use of the notation $f(x)$, which usually denotes a function. Therefore, it is more precise to refer to $f(x)=2x^3+ax^2-11x+6$ as a function rather than an equation.

Furthermore, the problem in Item 5 needs minor rewording. First, it is important to clearly state what the letter c in the equation represents. Second, the instruction for the first part of the question signals the item writer's intent of getting pupils to use the "*completing the square*" method for finding the x that gives a true statement when substituted into the quadratic equation, as well as to express x in terms of c. But the instruction contains these two mistakes: (i) "solve x" should be rewritten as either "solve for x" or "solve the equation", and (ii) the word "squares" should take the singular form. Likewise, the next part of the question contains another mistake other than the wrong use of the term "expression". The suffix s in the word "value(s)" should not be in parentheses as the term "range" suggests that there is more than one possible numerical quantity for c.

Apart from the word "equation", another case of misusing mathematical term occurs in Item 6. The four expressions in parts (a) to (d) have been described as quadratic, but upon closer examination, only the first two are found to be true while the remaining two fail to qualify as quadratic because they do not satisfy the general form ax^2+bx+c, where a, b and c are constants and $a\neq0$. Rather, $9mn+12m+6n+8$ and $ab-2ad-3cb+6cd$ can be regarded as linear algebraic expressions (Ministry of Education, 2006). For instance, $9mn+12m+6n+8$ is a linear expression in n with constant m. Another manifestation of misusing the term "quadratic" is demonstrated in Item 8. The equation in part (b), $\frac{1}{y} - \frac{2}{y+1} = 3$, is not a quadratic equation but a fractional equation that can be reduced to a quadratic equation when simplified.

Case 3: Incorrect mathematics

Item 9


Given that $v^2 = u^2 + 2as$,
(a) find the value of v when u = 6, a = 2 and s = 16,
(b) express s in terms of v, u and a.

$v^2=u^2+2as$
$v^2=(6\times6)+(2\times2\times16)$
$v^2=36+64$
$v^2=100$
$v=10$

(b) $v^2=u^2+2as$
$v^2-u^2=2as$
$\frac{(v^2-u^2)}{2a}=s$

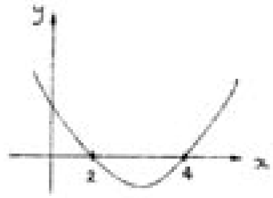
(a) 10 (2) 2
(b) $\frac{(v^2-u^2)}{2a}$ (2) 2

Item 9 seems like a typical, well-designed textbook or examination problem, but a flaw is detected in part (a) upon careful examination. To find the answer to this part, it entails solving a simple quadratic equation of the form $v^2=k$, where k is a positive value obtained by substituting the given values of u, a and s into the formula. Clearly, the equation will yield two distinct values of v, and hence this part question is flawed when the suffix s is omitted in the word "value". Such a mistake may be attributable to human carelessness in typing the question, but the evidence of awarding full marks to a partially correct answer, as shown in the student's solution provided herein, seems to indicate a weakness in the teachers' concept of square roots.

Case 4: Insufficient details

Item 10

Given the following sketch of a quadratic curve,



(a) Find the equation of the quadratic curve and state its factors clearly. (2m)
(b) Use factor theorem to show that the factors you have found in part (a) are correct. (3m)
(c) Find the remainder using the remainder theorem when $f(x)$ is divided by $(x-3)$. (1m)

Item 10, testing pupils on a few Additional Mathematics concepts, contains a few flaws. The one in part (a) may be less obvious than the others because when a group of student teachers was asked to do this item, only one of them was able to identify it. But such a flaw occurs frequently and is therefore worth noting. When the y-intercept of the quadratic function is not clearly specified, part (a) then becomes an open-ended question, with several possible answers for the equation of the function. Notably, the function $y=k(x-2)(x-4)$, where k is any positive real number, is a case in point because its graph will resemble the one in the sketch. However, this situation will not arise if the y-intercept is stated. As for part (b), the rationale for having this question seems to lack a clear and meaningful direction in view of the fact that the factor theorem could have already been applied to find the factors in part (a). For instance, a pupil may draw on the information that $x=2$ when $y=0$ from the sketch to derive the factor $(x-2)$. Thus the item writer should have constructed a more purposeful task than the original one to test pupils on the factor theorem. Besides the weak rationale, the number of marks allocated for parts (a) and (b) also seems disproportional, considering the amount of work required to be done in both parts. Consequently, part (b) appears to reward pupils who are able to obtain the equation of the quadratic function in the factorised form generously, and on the other hand, unfairly penalise those who are unable to do it as a result of not getting the equation. Finally, the notation $f(x)$ is mentioned in part (c), yet there is no reference in the sketch or the stem to make explicit what it actually represents.

Conclusion

This article outlined four types of flaws in algebra test items and aimed to encourage mathematics teachers to eliminate these flaws from their test items to provide well-constructed and good quality problems for pupils. The flaws can be corrected by pilot testing the items on a fellow colleague (Dindyal, 2006) or getting an experienced colleague such as the head of department, senior teacher or level head to carefully vet the items. Despite having these measures in place, flaws can still slip into test papers if the item writers and reviewers are uncertain about the correct way to phrase test items or the proper usage of certain mathematical jargons. To overcome these situations, they can check out questions in printed materials such as textbooks and past years' O and N levels examination papers or even seek professional advice from mathematics specialists.

In conclusion, setting well-designed test items is very important because they will produce valid test results that provide accurate information about pupils' learning. This information will in turn enable teachers not only to ascertain what pupils know and understand, but also to identify areas of deficiency in need of remediation or further learning. Consequently, teachers should excercise caution when designing test items.

References

Dindyal, J. (2006). Defective assessment items in mathematics. *Maths Buzz, 7*(2), 6 – 7.

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On Polynomials, Remainder and Factor Theorem In the Additional Mathematics Curriculum

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This note is extracted from a segment of a content upgrading in-service workshop for Additional mathematics teachers conducted in 2006.

A survey of some past year examination questions on Polynomials, Remainder and Factor Theorem reveals that the focus of the Additional Mathematics topic on polynomials is almost exclusively on the application of the remainder and factor theorem, and the consequent application of the factor theorem to find the roots of the cubic equations. There is little evidence in stressing the importance of developing pupils' conceptual understanding of multiplication and division of polynomials, and, in particular the division algorithm of polynomials. As assessment drives the curriculum, it is understandable that such concepts are not emphasised in classroom teaching.

For example, consider the following problem on polynomials.
The remainder when a polynomial f(x) is divided by x – 2 and x + 3 are 2 and 3 respectively. Find the remainder when the same polynomial is divided by the polynomial (x – 2)(x+3).

A quick search in the internet shows that questions of the above type, which requires division algorithm on top of the remainder theorem, are common difficult questions among high school students.

In this note, we shall discuss issues on (i) division algorithm as being the more "basic" than the remainder and factor theorem; (ii) remainder and factor theorem, and (ii) one way of identifying the possible first rational root of a cubic (or higher degree) polynomial.

1. Division Algorithm of Polynomials

To allow pupils to have a more in-depth understanding of polynomial arithmetic (in analogy to arithmetic of numbers), multiplication and division of polynomials should not be skipped in favour of the remainder and factor theorems and their applications in typical examination questions.

According to division algorithm of polynomials, it can be shown[1] that given any two polynomials P(x) and D(x), there exist unique polynomials Q(x) and R(x), where degree of R(x) is less than degree of D(x), such that the identity

$$P(x) \equiv Q(x)D(x) + R(x)$$

holds true for all real values of x. This is the direct motivation from the well-known division algorithm (manifested as long division) for numbers. Analogous to the long division for numbers, Q(x) is called the quotient and R(x) the remainder when the polynomial P(x) is divided by D(x).

Given the polynomials P(x) and D(x), the quotient and remainder when P(x) is divided by D(x) can be found directly by long division or by equating polynomials and then comparing coefficients of the different terms (see [**1**, Chapter 19, Examples 1 to 2]).

Here three points about Division Algorithm of Polynomials are mentioned below:

a) **Long division** The method of long division provides the pupils with a set of algorithm, which is analogous to the long division algorithm for integers, to systematically obtain the quotient and remainder from a division process.

b) **Equating equivalent polynomials** On the other hand, by equating two equivalent polynomials and then comparing coefficients allows pupils to strengthen their algebraic manipulation skills, which is also crucial for the new topic on Partial Fractions introduced into the new Additional Mathematics. The second method of comparing coefficients requires pupils to have a more conscious application of

(Footnotes)

[1] Pupils need not know the proof of division algorithm. This concept of division algorithm is parallel to the concept of division algorithm of numbers (or long division) which the pupils have already accepted since their primary school days in the mathematics curriculum.

the division algorithm more frequently, compared to the first method of long division. With the relative merit of the individual methods in mind, teachers may use their professional judgment to decide which method to teach their pupils, taking into account many other external factors. The detail of the two processes is discussed in the reference [**1**, Chapter 19] and will not be elaborated here.

c) **Deducing quotient and remainder from polynomial identity** While it is important to teach the procedures of finding quotients and remainders, the ability to recognize quotient and remainder from a given identity should not be overlooked. For examples, questions like the following, which do not require tedious computations but more of identification of the corresponding terms, could be done with pupils.

> Q1. It is given that
>
> $P(x) \equiv (3x^2 + 5)(6x^3 - 7x + 1) + (3x^2 + 5) + 1.$
>
> Find the quotient and remainder when
>
> (a) $P(x)$ is divided by $6x^3 - 7x + 1$
> (b) $P(x)$ is divided by $3x^2 + 5$
> (c) $P(x)$ is divided by $(3x^2 + 5)(6x^3 - 7x + 1)$
> (d) $P(x) - 1$ is divided by $3x^2 + 5$.
>
> Q2. It is given that
>
> $P(x) \equiv (7x^2 + 5)(3x+6) + 3$
>
> $Q(x) \equiv (x^3 - x^2 + x)(x + 2) + 7$
>
> Find the quotient and remainder when
>
> (a) $P(x)$ is divided by $7x^2 + 5$
> (b) $P(x)$ is divided by $x + 2$
> (c) $Q(x)$ is divided by $x^3 - x^2 + x$
> (d) $Q(x)$ is divided by x
> (e) $P(x) + Q(x)$ is divided by $x + 2$.

Thus, in covering division algorithm, while it is important to teach the procedures of division algorithm, the gist of the algorithm should not be replaced by the tedious processes of long division or comparing coefficients of two equivalent polynomials. The use of polynomial identities to identify quotients and remainders can indeed be used to sharpen pupils' accuracy in identification and further reinforce the concept of division algorithm.

2. Remainder and Factor Theorem

Remainder theorem can be seen as a special application of division algorithm. When a polynomial $P(x)$ is divided by $D(x)$ in the special case when $D(x)$ is of degree one (i.e. $D(x)$ is of the form $ax+b$), then its remainder is of degree less than one, which has to be a constant r. By using the division algorithm and substitution of a suitable value, it can easily be seen that the remainder $r = P\left(-\frac{b}{a}\right)$ (see [**1**, P127]). This is the well-known result of the Remainder Theorem in Polynomial Algebra.

It should be stressed that the remainder theorem, being only a special case of Division Algorithm, is restricted at least in the following sense:

a) the remainder theorem can only be used to find the ***remainder*** only – no statement is made at all in the remainder theorem about ways to find the quotient of the division of polynomials; and

b) the remainder theorem only works when the divisor is of ***degree one***; when the divisor is of degree higher than one, then one will have to revert to the division algorithm to find the quotients and the remainder of polynomial divisions.

Thus it is not sufficient to emphasise almost exclusively on Remainder Theorem without at least an equal emphasis on Division Algorithm of polynomials.

The factor theorem says, deriving from division algorithm and remainder theorem, that a polynomial $P(x)$ has a factor $ax+b$ if and only if $P\left(-\frac{b}{a}\right) = 0$. The relation between division algorithm, remainder theorem and the factor theorem can be summarized as shown.

> **Division Algorithm**:
> Given two polynomials $P(x)$ and $D(x)$, there exist unique polynomials $Q(x)$ and $R(x)$ where degree of $R(x)$ is less than degree of $D(x)$, such that $P(x) = Q(x)D(x)+R(x)$.
>
> Special case when $D(x)$ is linear
>
> **Remainder Theorem**:
> When a polynomial $P(x)$ is divided by $ax+b$, then its remainder can be written as $P\left(-\frac{b}{a}\right)$.
>
> **Factor Theorem**
> The term $ax+b$ is a factor of $P(x)$ if and only if $P\left(-\frac{b}{a}\right)=0$.

It is useful to recognize the relation among the three: division algorithm, remainder theorem and factor theorem. Some personal experience through my contact with trainee teachers and some secondary school teachers shows that considerable school students could not distinguish between the statements of remainder theorem and factor theorem, even though most of them have no difficulty in solving typical examination problems.

Suggested Activity

While teaching division algorithm, remainder and factor theorem, it is possible to get the pupils involved in more in-depth thinking about results related to these theorems. Sample of true/false questions as shown below can be used for the pupils.

Worksheet on Polynomials, Remainder and Factor Theorem

Decide on whether each of the following statements is **TRUE** or **FALSE**.

1. The remainder when f(x) is divided by $ax+b$ is the same as when the polynomial f(x) is divided by $x+\frac{b}{a}$.
TRUE / FALSE

2. The remainder when f(x) is divided by $x^2 + 2x + 3$ can be expressed in the form $\mathcal{A}x + \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are numbers.
TRUE / FALSE

3. The remainder when f(x) is divided by x can be found by f(1).
TRUE / FALSE

4. The remainder when f(x) is divided by x equals the constant term of the polynomial.
TRUE / FALSE

5. Given that $x^2 - 3x + 2 = (x-2)(x-1)$, we can conclude that the quotient when x^2-3x+2 is divided by $x - 2$ is $x - 1$.
TRUE / FALSE

6. When a cubic polynomial $ax^3 + bx^2 + cx + d$ is divided by a polynomial $ex^2 + fx + g$, its remainder can be obtained from the remainder theorem.
TRUE / FALSE

7. When a polynomial P(x) is divided by D(x), its quotient is Q(x) and remainder is R(x). So when P(x) is divided by Q(x), its remainder is R(x) and its quotient is D(x).
TRUE / FALSE

3. Identifying the first rational root of a polynomial equation

Given a cubic (or higher degree) polynomial, the usual procedure taught in the Additional Mathematics curriculum in solving the cubic equation is to obtain the first rational root by trial-and-error method. The standard procedure of trying the roots are ±1, ±2, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$ and so on and so forth. It would be disastrous if the first root does not lie within "easy reach" of this approach (for instance, what if the only rational root of the polynomial equation is $\frac{1}{37}$?).

There is a theorem in the undergraduate mathematics, classified under Undergraduate Algebra, which could be useful for this purpose of identifying the first rational root of a cubic equation. The theorem can be stated as follows:

Theorem. Consider the polynomial equation $a_0+a_1x+a_2x^2+\ldots..+a_nx^n=0$. Let us assume that all the coefficients $a_0, a_1,\ldots\ldots,a_n$ are integers. Then if a rational number $\frac{p}{q}$ (assume that p and q have no common factor, i.e. in its simplest form) is a root of the polynomial equation, then a_0 must be divisible by p and a_n must be divisible by q.

The proof of the theorem is not difficult. Any pupil with a strong sense of numbers or some knowledge of number theory can immediately see why this theorem is true. Nevertheless, its application in locating the first rational root of a cubic (or higher degree polynomial) equation is useful and not too difficult. We shall illustrate the application of this theorem with three examples below.

Example 1. Given the polynomial equation $7x^2+6x-1=0$. If $\frac{p}{q}$ is a rational root, then 7 is divisible by q and -1 is divisible by p. Thus, the only possible choices of q are ±1 or ±7; the only possible choices of p are ± 1. Hence all the possible roots the equation can have are either ±1 or $\pm\frac{1}{7}$. ***There are no other possible rational roots that this equation may have.*** By checking, the roots are in fact $\frac{1}{7}$ and -1. However, the application of this theorem to a quadratic polynomial equation is uninteresting since most pupils know how to solve such equations by either trial-and-error factorization or completing squares.

Example 2. Consider the polynomial equation $121x^3-143x^2+23x-1=0$. If $\frac{p}{q}$ is a rational root, then 121 is divisible by q and -1 is divisible by p. Thus the possible choices of p are ± 1 and the possible choices of q are ±1, ±11 or ± 121. Thus all the possible choices of the rational roots of this equation are ± 1, $\pm\frac{1}{11}$ or $\pm\frac{1}{121}$. By direct checking, it can be seen that indeed $\frac{1}{11}$ is a root to the given equation. ***There are no other possible rational roots that this equation may have.***

Example 3. Consider the polynomial equation $x^3 + 121x + 1 = 0$. By using the theorem (in this example the detail steps in deriving the roots as in Examples 1 and 2 above are left for the readers to work out), the only possible roots are ± 1. However, it can easily be checked that both 1 and -1 are not the roots of the equation. Thus, this equation does not have any rational roots[2].

Thus the above theorem can provide a more systematic procedure to aid in finding the first rational root of a polynomial equation.

4. Conclusion

In conclusion, we have discussed three main issues pertaining to the teaching of polynomials, remainder & factor theorem:

1. The teaching of division algorithm is important in helping pupils achieve a more balanced view of the arithmetic operations on polynomials.

(Footnotes)

[2] Here, we remind the readers that any cubic equation will always have at least one real root. This is obvious if one is familiar with all the possible shapes of a cubic polynomial; the cubic curve always cuts the x-axis in at least one point. However, a polynomial equation may not have rational roots at all (as illustrated by this Example 3).

2. Remainder theorem can be seen as a special result of the division algorithm; it is restricted in the sense that remainder theorem is only for finding remainder when the divisor is linear. Factor theorem is a special application of division algorithm and remainder theorem.

3. A special theorem in undergraduate algebra can provide a systematic procedure to find the first rational root of a polynomial equation. Teacher can consider using the theorem to provide a more systematic way of locating the first rational root of a polynomial equation.

In line with the new Additional Mathematics syllabus which stresses more on conceptual understanding, more of the above discussion will be relevant and useful when incorporated into the teaching of this particular topic.

Reference

[1] Toh T.L., Algebra Resources for Secondary School Teachers, McGraw-Hill (Singapore), 2006.

The Parry And Queenie Problems

Daniel Hue and Marcus Yip

PROBLEM ONE: Introducing Parry and Queenie

Once upon a time, in the land of NIE, there existed one boy, Parry and one girl, named Queenie. These two *odd* people fell in love with each other which was fortunate because no one else can tolerate their obsession with prime numbers.

One day, I met them at *Prime* Tower Cafeteria in NIE library. Knowing that I'm a math student, they posed me two problems. If I succeed, they would treat me to *prime* ribs and a cup of *prime* roast coffee! So Parry (P) and Queenie (Q) came up with this ingenious riddle.

P: 7 years ago I was at odds with Q.
2 years ago I was in a confused state of adolescence. Was I the power of a prime or the power of a non-prime? Perhaps I am both.
This year, I came even with my age.

Q: 9 years ago I was in *primary* school!
8 years ago I was surer than P about my age. I was the power of a prime.
Last year, I felt really odd about my age.

1.1 What are the possible ages of P and Q respectively?

Parry and Queenie are "*kiasu*" parents in the making. They dream of having a child that would lead the land of NIE. Instead of trusting fortune tellers, they chose to determine their fates via *prime* numbers.

To ensure that their child will be the future leader, they think they must have their child when the **absolute difference between the products of the digits in their ages is a prime number.** To illustrate, suppose Parry is 23 and Queenie is 21. The products of the digits in their ages are 6 (2 times 3) and 2 (2 times 1) respectively. The absolute difference is |6-2| = 4, which is not a prime number. This means that they must not have their child when Parry's age is 23 and Queenie's age is 21.

2.2 If Parry and Queenie want to ensure that their next child turn out to be a future leader, based on their belief at what age must Parry and Queenie be when they have their child? Queenie does not want to give birth until next year.

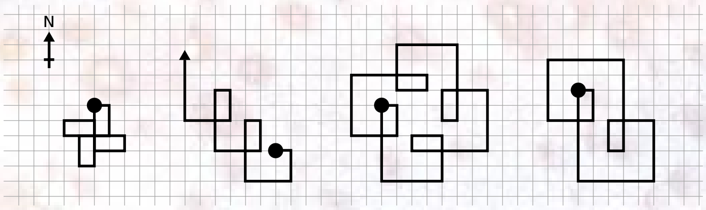
(Hint: Use the ages found in **1.1**. *Assume Parry and Queenie have the same date of birth: 1 January.)*

PROBLEM TWO: Spironames for Parry and Queenie's child

Parry and Queenie are expecting their first baby boy. But they need to come up with a name for their boy. Parry who loves mathematics and art, comes across this geometric design called 'spirolaterals', or spiros for short. Parry suggests to Queenie that they should use spironames.

What are spiros? [invented in 1973 by Frank C. Odds]

1. Start with a number sequence, e.g. 123. This is called the spirocode.
2. On a piece of graph paper, mark a starting point
3. Take the top of the paper as North, and move 1 unit to the East (right).
4. Turn and move 2 units South (down).
5. Turn and move 3 units West (left).
6. (Because we have run out of digits, start again at 1.)
7. Turn and move 1 unit North (up).
8. Keep turning 90 degrees clockwise: East, South, West, North, East, South, West, North.
9. Observe what happens. You are drawing a spiro!

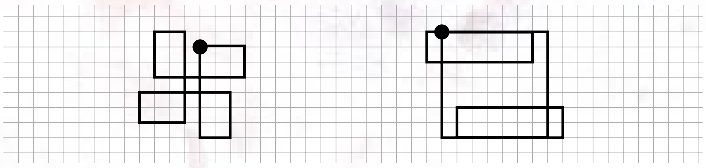


123 1234 12345 123456

What is a spironame?

First, use the alphanumeric keypad on a phone to convert your name into spirocode: A, B or C=2, DEF=3, GHI=4, JKL=5, MNO=6, PQRS=7, TUV=8 and WXYZ=9.

For example, in spirocode, DAN=326 and MARCUS=627287. Next, take your spirocode and follow steps 1-9 to draw the spiro. This is your **spironame**. Here are two examples of the spironames:



DAN [326] *(repeats after 4 cycles)*

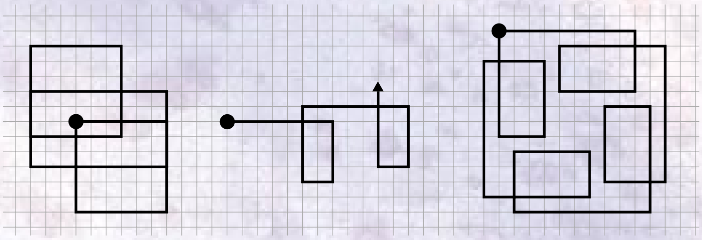
Marcus[627287] *(repeats after 2 cycles)*

Problems for Primary and Lower Secondary students

2.1 Draw your own spironame. Share your spironame with members of your group.

2.2 Look at your group members' spironames. Whose spiros repeat after a few cycles? How many cycles are required to repeat? Whose spiros never repeat? Can you spot a pattern?

2.3 Can you guess which mathematicians have these spironames?



Mathematician #1 Mathematician #2 Mathematician #3

[Hints: Mathematician #1 was one of the inventors of calculus. Mathematician #2 discovered this formula: Area = I + B/2 – 1. Mathematician #3 proved Fermat's Last Theorem in 1995.]

2.4 Parry and Queenie are looking for a spironame for their baby boy. They want the spiro to be easy to write, and to repeat after two cycles. Can you suggest 3 male spironames which repeat after two cycles? Draw the 3 spiros.

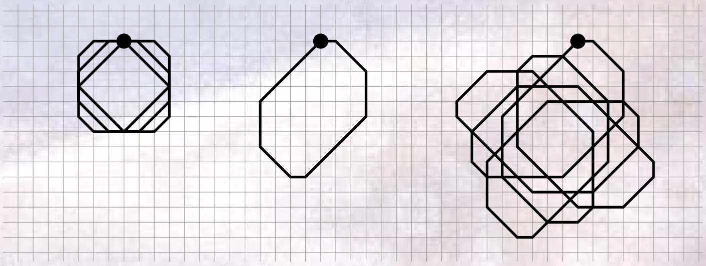
2.5 The doctor told Parry and Queenie that they will be having twins instead – a baby boy and a baby girl! Parry and Queenie want a beautiful spiro for their baby girl as welll. Can you find a female name with a beautiful spiro? Draw the spiro. Be as creative as you can!

Problems for Upper Secondary and JC students

3.1 Can you classify all spirocodes according to the characteristics of the spiros they generate?
*[Hint: For example, which spiros repeat? Which do not repeat? What criteria must the generating spirocode satisfy? When spiros repeat, how many cycles are required? E.g. the spirocode **326** takes 4 cycles to repeat, **627287** takes 2 cycles to repeat. Can you list all the possible number of cycles? What criteria must be satisfied for each number of cycles?]*

3.2 If we do not mark the starting point, does each spiro pattern have a unique generating spirocode? Give a proof or counterexample. *[Hint: Try 112 and 211.]*

3.3 A 45° spiro is a special type of spiro, in which we turn 45° clockwise each time. When moving diagonally, we treat the diagonal of the small square as one unit. Below are 3 examples of 45° spiros. Can you classify all 45° spirocodes according to the characteristics of the 45° spiros they generate?



123 (8 cycles) 1234 (2 cycles) 12345 (8 cycles)

Explanation for 123: For the number sequence 123, we move 1 unit East, 2 diagonal units SE, 3 units S, 1 unit SW, 2 units W, 3 units NW, 1 unit N, 2 units NE, 3 units E, and so on. Each time, we move from one lattice point to another. The spiro returns to its starting point after 8 cycles.

3.4 If you are good in programming, can you write a computer program to draw spiros?

Brief Solutions And Hints To The Parry And Queenie Problems

1.1 P=18, Q=16

1.2 (P,Q) = (19,17); (21,19); (41,39); (51,49); (71,69) [They adopt in the last two cases]

2.1 For example, see spironames of Dan and Marcus under 'What is a spironame?'

2.2 Dan and Marcus's spiros repeat after 4 and 2 cycles respectively. Mathematician #2's spiros never repeats. The pattern is discussed in **3.1** below.

2.3 Newton, Pick, Wiles

2.4 E.g. Ab, Andrew, Washington. (Any name with number of letters congruent to 2 mod 4.) More 2-letter, 6-letter, 10-letter names available at www.baby-boy-names.org

2.5 Leave this to the students' creativity. Ask the class to vote for their favourite spiro.

3.1 If the generating spirocode $s_1s_2s_3 \ldots s_L$ has length L, classify by cycles needed to repeat:

Cycle	Criteria	Examples
1	$L \equiv 0 \bmod 4$, $\Sigma s_{4i+1} = \Sigma s_{4i+3}$ and $\Sigma s_{4i+2} = \Sigma s_{4i}$	6464 ('MING')
2	$L \equiv 2 \bmod 4$	627287 ('MARCUS')
4	$L \equiv 1$ or $3 \bmod 4$	236 ('BEN'), 78272 ('SUBRA')
Infinite (Doesn't repeat)	$L \equiv 0 \bmod 4$, $\Sigma s_{4i+1} \neq \Sigma s_{4i+3}$ or $\Sigma s_{4i+2} \neq \Sigma s_{4i}$	7425 ('PICK')

[Hint: $\Sigma s_{4i+1} = \Sigma s_{4i+3}$ means 'Eastward moves = Westward moves', while $\Sigma s_{4i+2} = \Sigma s_{4i}$ means 'Southward moves = Northward moves'. E.g. if $L \equiv 1 \bmod 4$, each iteration of the spirocode changes the starting direction by a right angle, and it takes 4 cycles to return to the original starting direction. A proof can be achieved using vectors.]

3.2 Counterexample: 112 and 211 generate the same spiro pattern.

3.3 Let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $v_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $v_5 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $v_6 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $v_7 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_8 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_k = v_{k-8}$ for $k > 8$.
If the generating spirocode $s_1 s_2 s_3 \ldots s_L$ has length L, classify by cycles needed to repeat:

Cycle	Criteria	Examples
1	$L \equiv 0 \bmod 8$, and $\sum_{i=1}^{L} s_i v_i = 0$	12121212
2	$L \equiv 4 \bmod 8$	1234, 123456781234
4	$L \equiv 2$ or $6 \bmod 8$	12, 123456, 1234567812
8	$L \equiv 1, 3, 5$ or $7 \bmod 8$	1, 123, 12345, 1234567
Infinite (Doesn't repeat)	$L \equiv 0 \bmod 8$, and $\sum_{i=1}^{L} s_i v_i \neq 0$	12345678

3.4 An exercise for students familiar with programming and computer graphics. If students succeed, offer them an opportunity to present their work to the class.

About the writers
The writers, Daniel Hue and Marcus Yip (PGDE '06-'07) created the Parry and Queenie problems as part of the Teaching and Learning of Mathematics module tutored by Mrs Tan-Foo Kum Fong. For complete solutions, please write to the editor of Maths Buzz.

References. Part of problem 1.1 was modified from Coolmath.com, retrieved on 19 August 2006 from http://www.coolmath.com/math_puzzles/a2-age.html. Initial description of spirolaterals adapted from Sobel, Max & Maletsky, Evan. (1999) Teaching Mathematics: a sourcebook of aids, activities, and strategies (3rd edition), pp. 241-243. MA: Allyn & Bacon. Our extensions are original.

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