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Maths Buzz

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Contents

Contributions Invited

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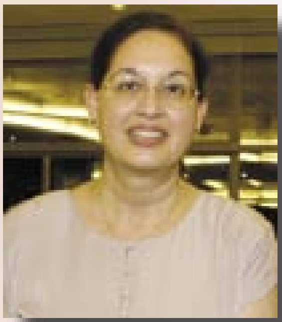
President's Message...



Mathematics Teachers Conference 2007
Theme: Mathematical Literacy
1st June 2007
National Institute of Education

Dear Mathematics Teachers,

This year we are once again having our annual mathematics teachers conference. The theme of our conference is Mathematical Literacy. Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of an individual's life as a constructive, concerned and reflective citizen. It is thus important that the teaching of mathematics go beyond the mere acquisition of concepts and skills in vacuum capsules. We must endeavour to facilitate the learning of mathematics amongst our students in ways that make them mathematically literate thus preparing them to cope well with demands in later life.

The conference will provide participants a day of acquiring knowledge, questioning practices and deliberating on issues of mathematical literacy. A special treat at the conference are the guest lectures by Prof Jin Akiyama from Japan, a well known personality who makes mathematics both fun and meaningful. Mathematics Educators and Mathematicians will deliver lectures and conduct workshops. More than 800 participants have registered for the conference. My committee and I look forward to meeting you at the conference.

A/P Berinderjeet Kaur

President
Association of Mathematics Educators
Nanyang Technological University

Tortoise and Rabbit with the 'second' hand

Tan-Foo Kum Fong
Geylang Methodist School (Secondary)
Mohammad Hafiz bin Mohd Rashib
National Institute of Education

There are many ways to teach mathematics effectively. The use of concrete manipulative to teach mathematical concepts is one approach, while the use of literature is another approach. Children's literature can also provide the context to help learners relate mathematics to real-life situation. This is important as it makes learning meaningful and provides the contexts for them to explore and develop ideas. Some children's literature makes explicit the mathematical concepts, for example, counting books and books on shapes. Other children's books might not be called 'math books' by teachers but yet offer rich possibilities of extended mathematical activities.

While there is a vast array of such commercially produced children's literatur,e teachers can also be the creators of children's literature which develops mathematics concepts meaningfully. Mohammad Hafiz bin Mohd Rashid adapted the familiar children's tale, ***The Tortorise and the Rabbit*** to introduce the concept of 'seconds' for young children. This is his creation.

Tortoise is helping Rabbit with his training for an upcoming race. Everyday, Tortoise uses a clock to time how long Rabbit takes to complete each run. One morning, Tortoise said to his friend: "Rabbit, you left the starting point at 10.30am and you ran past the finishing line at 10.32am."

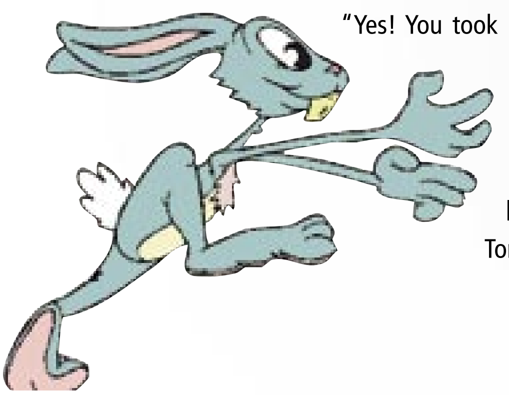
"Are you saying that I took 2 minutes to complete the race?" the Rabbit asked.

"Yes! You took 2 minutes. You need to run faster if you want to beat the other runners to the finishing line" replied the Tortoise.

Rabbit was determined to win. So he trained everyday.

One week later, Tortoise timed Rabbit in his run. "Rabbit, you left the starting point at 9.00am and you ran past the finishing line at 9.01am."

"Are you saying that I took 1 minute to complete the race?" the Rabbit asked.



"Yes! You took 1 minute. You need to run faster if you want to beat the other runners to the finishing line" replied the Tortoise.

Rabbit was determined to win and he trained even harder everyday. He ran and he ran. One week later, Tortoise timed Rabbit in his training and shouted excitedly, "Rabbit, you left the starting point at 9.45am and you ran past the finishing line at 9.45am."

"Are you saying that I took zero minute to complete the race?" the Rabbit asked.

"No! It's impossible to complete the race in zero minutes. You took less than 1 minute to complete the race," explained the Tortoise.

Rabbit was surprised and asked, "Less than 1 minute? How long is that?"

Tortoise took a long thin stick and attached one end of it to the centre of the clock and the other end pointed at the numbers on the clock face. The stick started moving around the clock straightaway. "Tick! Tick! Tick!".

Tortoise remarked: "Remembered in the previous race with me, Rabbit you fell asleep under the tree. As a result, I won the race. Since you came in second in that race, I will name this new hand on the clock the 'Second Hand'.

Next day, Rabbit ran hard in his training and this time, Tortoise kept counting to the ticking of the Second Hand.

"1 second, 2 seconds, 3 seconds, ….., 49 seconds, 50seconds, Stop!" shouted the Tortoise as Rabbit ran past the finishing line.

"Are you saying that I took 50 seconds to complete the race? Is that less than 1 minute?" asked Rabbit who was trying to catch his breath.

"Yes! You took 50 seconds which is certainly less than 1 minute. Maintain this timing and you have a good chance of winning the race," assured Tortoise.

Rabbit was so happy. He was determined to win and he trained even harder everyday till the day of the race. Rabbit ran and he ran.

During the race, Rabbit gave his best and won the race with a timing of 40 seconds. He broke the old record of 1 minute. Since then, Tortoise and Rabbit became very close friends. They no longer run competitively but thanks to them, we have a measure of time called 'seconds'.

Investigative problems in geometry and data handling

Lee Peng Yee
National Institute of Education

The trends in fashion are also applicable to mathematics. The latest trend in mathematics is about '*processes*'. The question here is how to learn and assess processes inherent in solving a mathematics problem. To help teachers answer that question, I conducted an in-service module on geometry and data handling for primary school teachers. The objective of the course is not to assess the geometric and data handling skills acquired but to provide platform for teachers to apply the skills learned and in the course of action discover the processes required to solve the problem. Each of the eight lessons consists of three activities and one investigative-type assignment- to assess the processes.

I have selected six assignments, five on geometry and one on data handling to be included in this article.

Problem 1 Congruence of quadrilaterals

We all know that three sides (SSS) determine a triangle whereas three angles do not (AAA). We also know that two sides and an included angle (SAS) determine a triangle, and so do two angles and an included side (ASA). In other words, SSS, SAS and ASA are the conditions that determine a triangle. *The task is to find conditions that determine a quadrilateral.*

Take, for example, the conditions SSSS (4 sides) or SASAS (3 sides and 2 included angles). Do these conditions determine a quadrilateral? What else? Investigate and write down the conditions that determine a quadrilateral and those that do not. If not, why not.

Problem 2 Cross-sections of a cube

Given a cube, and a plane cutting the cube, *the task is to describe the cross-sections obtained.*

For example, can the cross-sections be a triangle, an equilateral triangle, a square, or a trapezium? Note that two intersecting lines determine a plane. To display a cross-section, you need only to mark two lines on the cube to show where the cutting takes place. Name various shapes of the cross-sections, and how they are obtained by marking the two lines. Can a cross-section be a 5-sided or 6-sided polygon?

Problem 3 Volume of pyramids

We know that a cube can be cut into 3 equal pyramids, and we can construct a net for the pyramid, using the Pythagoras theorem. Now suppose we have a prism with a triangular base. *The task is to cut the prism into 3 pyramids (not identical) having the same volume, and construct a net for each pyramid.*

To make it more precisely, let the triangular base be the right-angled triangle with sides 3, 4, 5 cm and the height of the prism 6 cm. Construct the 3 pyramids one at a time. First, use the base of the prism as the base of the first pyramid. Construct a net for the pyramid. Then use the top of the prism as the base of the second pyramid. Construct a net for the second pyramid. Put them together and figure out the third pyramid.

Problem 4 Similar quadrilaterals

All circles are similar. Two triangles are similar if the AAA condition holds. Find conditions under which the following are similar.

(a) Two squares
(b) Two rectangles
(c) Two rhombuses
(d) Two parallelograms
(e) Two trapeziums
(f) Two kites

Observe the conditions above, and *the task is to find conditions under which two quadrilaterals are similar.* Give reasons, and make further comments, if any.

Problem 5 Graphical representation

Collect the 2 numbers in a lift, namely, the total number of persons and the total weight that the lift can carry. *The task is to collect data, to represent the data graphically, to interpret the graph, and to ask further questions.* In other words, answer the following questions.

(a) What information can you read from the collected data?
(b) How do you represent the data graphically?
(c) What can you read from the graph?
(d) What further information would you like to know?
(e) What additional data do you need to collect in order to find out more information?

Selected answers

Here are some selected answers to the problems above. It is hard to have standard answers for such problems. In fact, there is none. Even if there were, how the answers are presented matters too. In Problem 1, SSSS does not determine a quadrilateral, whereas SASAS does. Other conditions can be found by permuting five letters consisting of S and A. In Problem 2, we can obtain cross-sections that are a triangle, an equilateral triangle, a square, a trapezium, and more. We can obtain a 5-sided and a 6-sided polygon.

In Problem 3, the nets are given by the triangles (with sides in brackets and in cm) listed below.
The first pyramid: (3, 4, 5), (3, 6, $\sqrt{45}$), (4, 6, $\sqrt{52}$), (5, $\sqrt{45}$, $\sqrt{52}$).
The second pyramid: (3, 4, 5), (4, 6, $\sqrt{52}$), (5, 6, $\sqrt{62}$), (3, $\sqrt{52}$, $\sqrt{61}$).
The third pyramid: (3, 6, $\sqrt{45}$), (3, $\sqrt{52}$, $\sqrt{61}$), (5, 6, $\sqrt{61}$), (5, $\sqrt{45}$, $\sqrt{52}$).
It is instructional to construct physical models of the pyramids in order to see that they do have the same volume.

In Problem 4, two quadrilaterals are similar if the AAAA condition holds. So does the condition when four corresponding sides are proportional. To find further conditions, start with a condition used in Problem 1. Then try to relax the condition so that the two quadrilaterals are similar and not necessarily congruent.

In Problem 5, we can use the scatterplot to present the data. We can look into the ratio of the total weight over the number of passengers. To make the discussion more interesting, we should collect data from the lifts of various sizes and from different places such as housing estates, shopping centres or condos. We must always go one step further, and ask what is missing. Then discuss what we can do to find out the missing data.

Rubric

We may design a complicated rubric. However we may also design a simple one. Here are some suggestions. If you can obtain three pieces of information for a problem, it is satisfactory or Grade 2. If five or six pieces of information, you are good or Grade 3. If more than six pieces, you are very good or Grade 4. To deserve Grade 3, you should be able to explain clearly so that your fellow teachers understand. To deserve Grade 4, you must be able to write it in such a way that your fellow teachers appreciate it. It is important that you obtain some results. It is equally important if not more if you go through the process.

After having tried out the above problems, the next challenge is to set good investigative problems, and to set a good selection of such problems.

Difficulties with geometry proofs

Mr Leong Yew Hoong
National Institute of Education

Mathematical proofs have recently enjoyed greater attention among secondary mathematics teachers. This can be seen from the inclusion of "proofs in plane geometry" in the revised Additional Mathematics syllabus. In this article, I mused about some hypothetical difficulties some students may have when confronted with proofs and ways in which we can adapt our presentation of proofs to help them better grasp the idea of "proof".

1. "What is this thing (proof) all about? I don't get what the teacher is trying to do."

Because most of their earlier mathematical experiences are about "problems to *solve*", students are more used to "solving" – finding 'answers' (or conclusions). In the case of "problems to prove", however, conclusions are already given. In "proofs", instead of finding conclusions, the task is to argue why the conclusion 'makes sense' mathematically.

Expressed in pictorial form, in "problems to *solve*", the emphasis is on the conclusion:

Given conditions ⟶ **Conclusion**
Working **(getting the 'answer' right)**

In "problems to *prove*", the emphasis is on the reasoning in the working:

Given conditions ⟶ Conclusion
Working
(getting the 'step' right)

[Note the use of the word "emphasis" in differentiating both types of problems. In claiming that the emphasis in "problems to solve" lies in the conclusion, I am in no way agreeing that the process of reasoning towards the conclusion is not important. Rather, compared to "problems to proof", the shift in emphasis is relatively heavier towards the conclusion.]

A set of actual examples may illustrate the difference more clearly.

(I) A problem to solve: An interior angle of an isosceles triangle measures 50° . Find the possible sets of measures for the other interior angles of the triangle.

(II) A problem to prove: Prove that the base angles of an isosceles triangle are equal.

In problem (I), the aim of the solving process is to find the answers (conclusion):

<u>Given conditions</u> ⟶ **<u>Conclusion</u>**
Working
Isosceles Triangle — **Case 1: 50° and 80°**
Interior angle 50° — **Case 2: 65° and 65°**

In problem (II), the aim of the proving process is to provide the explanation (working) that links the condition to the conclusion:

<u>Given conditions</u> ⟶ <u>Conclusion</u>
Isosceles Triangle — **Working (Linking the condition to conclusion)** — Base angles are equal

If another example in 'real life' is helpful to depict the focus on process rather than the conclusion, a useful familiar metaphor of proving is that of presenting in a TV cooking show.

The ingredients of the cook-presenter are like the 'given condition';
The finished product is like the 'conclusion';
And the process of demonstrating the cooking is the working of the proof.

<u>Given conditions</u> ⟶ <u>Conclusion</u>
Ingredients — **Working (Demonstrating the process clearly)** — Cooked food

Like cooking demonstration, the emphasis in proofs is not the ingredient nor the finished product but the process. That is why we scrutinize the steps, ask questions and make sure that each step is correct.

Like a cook-presenter, we do not miss any steps or assume that the audience knows some hidden techniques. We assume they do not know and the job of the presenter is to make every step clear and explicit. Similarly, in proofs, it is important to make the steps explicit even though it may appear obvious to us.

2. "Why must prove what we already know?"

This question perhaps stems partially from a narrow view of mathematical "proofs". Some think of proofs merely as ways of "establishing truth". According to this view, when doing proofs, we are 'making sure it is correct'. While this function of proof is applicable in some cases, it certainly does not work very well in situations when students are already convinced of the truth of a theorem before even proving it.

Example:

Teacher: Now let's prove that base angles of an isosceles triangle are equal...
Student: Huh? Why need to prove? We know this since primary school.
Teacher: By proving it, we make it very sure...
Student: I am very sure already. Anyway, anything you say (sigh)...

Another view of proofs in (especially secondary-level) mathematics is that it provides "<u>mathematical explanation</u>" of something we already know. It 'unearths' the geometrical relations which remains unknown without the attempt to prove. In other words, proofs not only answer the "is it true?" question, it also answers the "why is it true?" question as well.

Returning to the specific example of proving "base angles of an isosceles triangle are equal" may illustrate this better. Before we learn proofs, we only know the first and the third columns – the given condition and the conclusion:

Figure 1: Given condition and conclusion

But proofs 'fill in the gap' – by providing mathematical reasons – for 'why it is true':

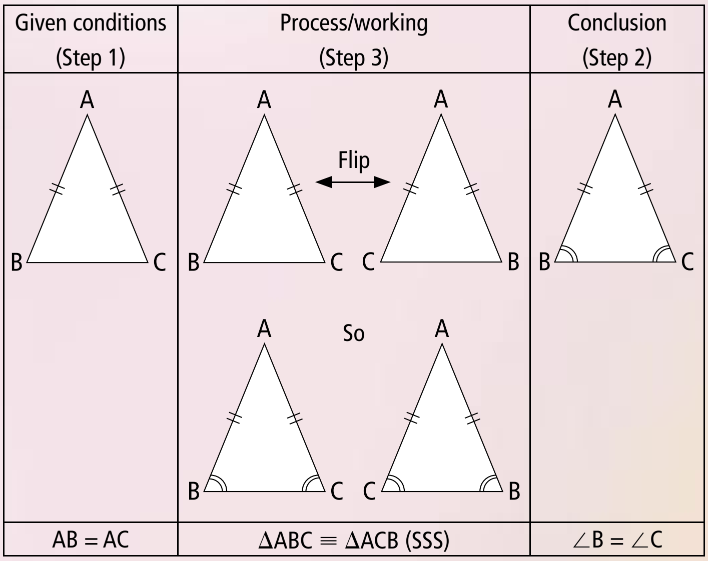


Figure 2: Condition, conclusion, and process (diagrammatic overview)

3. "Teacher showed me so many times already. But I still cannot write the proof on my own!!!"

Students who are new to proofs often find it very hard to write out a 'standard proof' even though the teachers might have shown numerous times the steps on the board. Teachers often interpret students' inability to write proofs as a total lack of understanding of the proof process. This may not necessarily be true. I suggest there are at least three levels of proof-proficiency:

(I) Diagrammatic overview of strategy;
(II) Translation to textual form;
(III) Inclusion of reasons for each step in the textual form.

Again, using the example on the proof of "base angles of an isosceles triangle are equal" may illustrate the above levels in a clearer way.

Figure 2 above gives an illustration of what level I proof-proficiency may look like. Level II requires the ability to translate the visual forms in figure 2 into textual forms:

Given conditions (Step 1)	Process/working (Step 3)	Conclusion (Step 2)
AB = AC	Consider ΔABC and ΔACB. AB = AC AC = AB BC = CB So, ΔABC ≡ ΔACB (SSS) Therefore, ∠B = ∠C	∠B = ∠C

When students are able to include geometric reasons to substantiate the claims in each step of the working, then it demonstrates the operation of level III, as shown below.

Consider ΔABC and ΔACB.
AB = AC **(given condition)**
AC = AB **(given condition)**
BC = CB **(same side)**
So, ΔABC ≡ ΔACB **(SSS)**
Therefore, ∠B = ∠C **(all corresponding angles are equal in congruent triangles)**

Some students may not be able to reach level III immediately. It may help if teachers progress slowly 'through the levels'. Conventional instruction tends to bypass level I altogether. There is perhaps a need to take into account the need to devote some time to level I type of activities in the teaching of geometric proofs.

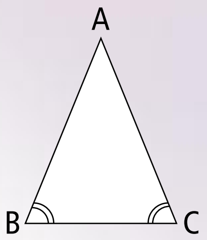
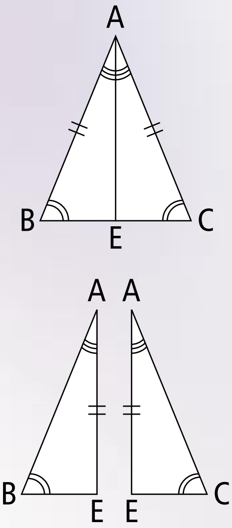
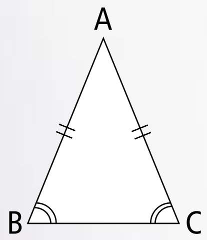
A sample template

I end this article by suggesting a possible sequence and accompanying template that can be used to teach geometric proofs that takes into account the gradual movement through the levels.

Problem: If two angles of a triangle are equal, then the sides opposite to these angles are equal.

I proceed in this order:
(1) Write down clearly the (a) "given condition" and the (b) "conclusion";
(2) Present the diagrammatic overview in the spaces provided;
(3) Translate the diagrams into textual arguments in the next row below (2) above; and
(4) Fill in the reasons for each step in (3) above in parentheses.

The above steps can be presented in their respective steps using the templates shown below.

(1a)Given condition	(2) Diagrammatic overview	(1b)Conclusion
		
∠B = ∠C	By AAS	AB = AC

(3) Textual arguments	(4) Reasons
Insert AE as the bisector of ∠A.	
Consider ΔABE and ΔACE.	
∠B = ∠C	(given condition)
∠BAE = ∠CAE	(AE bisects angle A)
AE = AE	(common side)
So ΔABE ≡ ΔACE	(AAS)
Therefore AB = AC	(all corresponding sides are equal in congruent triangles).

An earlier version of this article first appeared in a handout for an in-service course for mathematics teachers entitled "Geometry in secondary Additional Mathematics" held at Swiss Cottage Secondary in Jun 2006. I thank the teachers for their participation and contribution. I have included things I learnt from them in this revised article.

Solving a Triangle: The Ambiguous Case

Jaguthsing Dindyal
National Institute of Education

One of the important aspects of trigonometry is "solving triangles", i.e. finding the measures of unknown sides and angles. Students generally use the Law of Sines or the Law of Cosines to solve triangles. Four cases can be identified:

1. Case 1: Three sides are given (SSS).
2. Case 2: Two sides and the included angle are given (SAS).
3. Case 3: Two sides and the angle opposite to one of them are given (SSA).
4. Case 4: One side and two angles formed with the side are given (ASA).

Case 1 and Case 2 are clear cases for the use of the Law of Cosines and Case 4 typically requires the use of the Law of Sines. However, Case 3 is problematic. In this case, the use of the Law of Sines to determine the missing measures of a triangle can give rise to three possibilities: (1) no triangle may exist, (2) only one triangle may exist, and (3) two distinct triangles may exist. Hence, when two sides and the angle opposite to one of them are given (SSA), there is an ambiguity as there are three different possibilities.

Secondary level mathematics syllabuses tend to avoid the ambiguous case as it is more demanding on students. Also, teachers have a difficulty with this topic, as the ideas related to this topic are not expressed clearly in textbooks or they are altogether absent. Given the importance of trigonometry in the school curriculum, the Ambiguous Case needs to be addressed properly for teachers and pupil alike.

Two different approaches may be used to explain the Ambiguous Case. The first approach is to use diagrams to illustrate the three possibilities (one triangle, two triangles or no triangles) when the SSA condition is provided. It is a visual approach and helps the individual students to grasp the essence of the SSA condition. The second approach is based on the use of the Law of Cosines and describes basically how the SSA condition can be used together with the Law of Cosines to relate to the various possibilities of the first approach. Using these two approaches and showing how they relate to one another, may provide a better understanding of the Ambiguous Case for solving triangles. However, in this paper only the first approach will be discussed.

Consider triangle ABC in which, using usual notations, angle A, a, and b are given. Let h be the height of the triangle. So, $h = b\sin A$

First Approach

1. If A is acute and $a < h$ (or $a < b\sin A$), then triangle ABC does not exist, as illustrated in Figure 1 below.

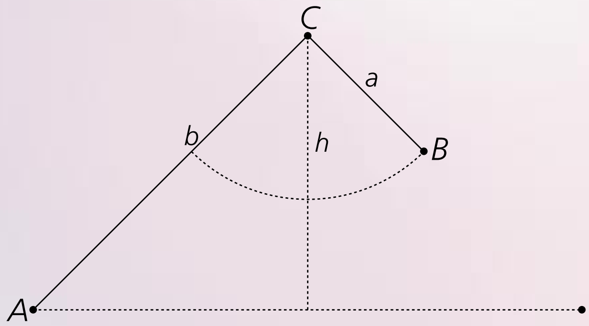


Figure 1

2. If angle A is acute and $a = h$ (or $a = b\sin A$), then only one triangle exists and this triangle is right-angled.

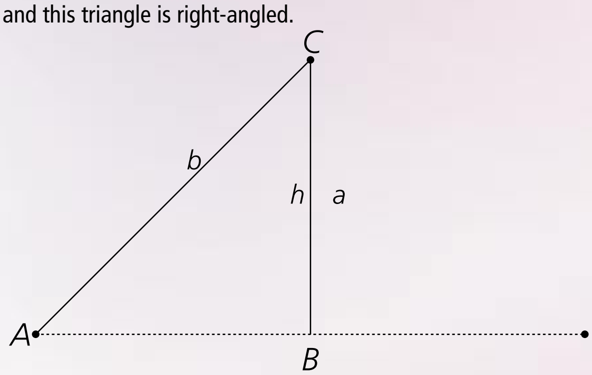


Figure 2

3. If angle A is acute and $h < a < b$ (or $b\sin A < a < b$) then two triangles exist when B takes the possible positions and, as illustrated in Figure 3 below.

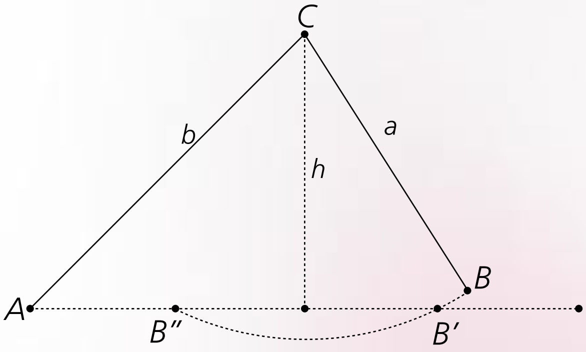


Figure 3

4. If angle A is acute and $a > h$ (or $a > b\sin A$) then only one triangle exists

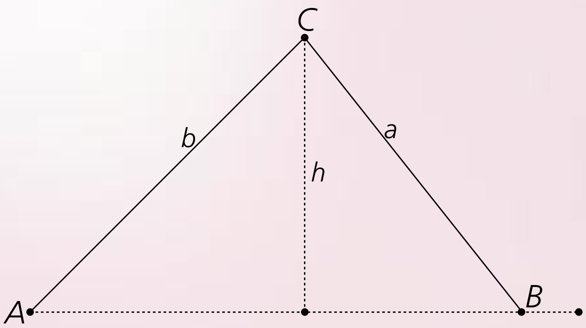


Figure 4

5. If A is obtuse and $a < b$ or $a = b$, then no triangle exists.

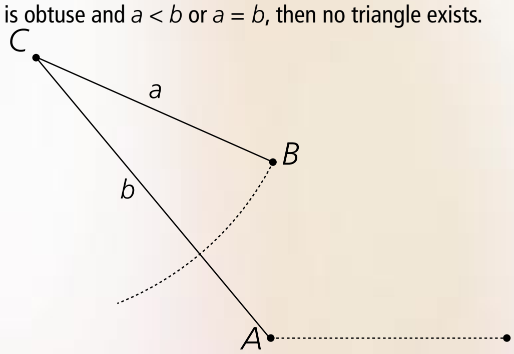


Figure 5

6. If angle A is obtuse and $a > b$ then one triangle exists.

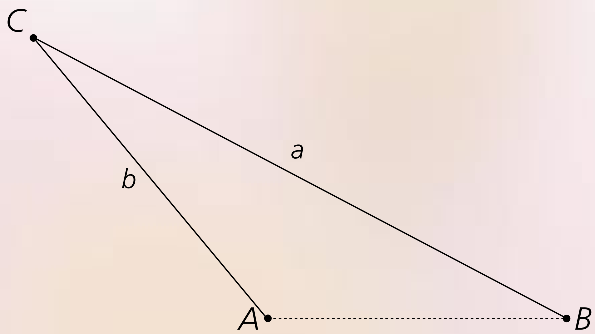


Figure 6

Visualising the Ambiguous Case is quite hard for many students. The above diagrams illustrate a visual aspect of the Ambiguous Case (Figures 1 to 6). Figure 1 and Figure 5 present the cases when a triangle does not exist whereas Figure 2, Figure 4, and Figure 6 present the cases when only one triangle exists. It is to be noted that Figure 2 represents the RHS case which is the only SSA case for congruency. Figure 3 illustrates the case when two triangles exist. It is expected that this visual aspect will provide a better understanding of why the SSA case is called the ambiguous case.

Squeeze Your Brain:

Frisky Frogs

Across a stream runs a row of seven stepping stones. On the first three stones (1st, 2nd and 3rd) sat three female frogs: Fergie, Francine, and Freda who wanted to get to the last three stones (5th, 6th and 7th in the same order) without getting wet.

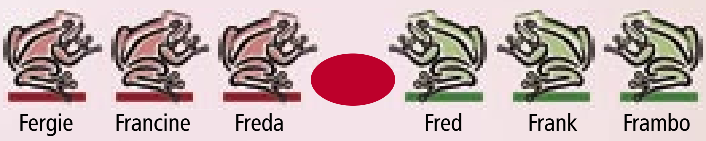
On the last three stones (5th, 6th and 7th) sat three male frogs: Fred, Frank, and Frambo, who were waiting to hop over to the first three stones (1st, 2nd and 3rd in the same order) without getting into the stream.

Only one frog is allowed to move at any one time. Besides, as the stones are rather small, only one frog can rest on any one stone.

To begin, the unoccupied stone is in the centre separating the female and the male frogs. How many moves will the frogs take to move over to the designated stones?

Can you come up with different ways of solving the problem?

If the number of frogs were increased to 8, 10 …and subsequently to 2n, is there a general pattern for the number of moves?



The Frisky Frog problem was adapted by Ms Wong Wai Mun and Ms Wong Wai See from the book, A. Hart-Davis, (1998). Amazing math puzzle. New York: Sterling Publishing.

Leapy Frog

A little green frog sits at the bottom of the stairs. He wants to get to the 10th step so he leaps up 2 steps but falls backward by 1 step. However, he continues leaping and for every 2 steps he leaps, he falls backward by 1 step.
How many leaps will he take top reach the (i) 10th step, (ii) 100th step (iii) n th step?

Extension:
2 frogs Freda and Frambo are having a leaping competition. Freda takes 3 steps forward but falls backward by 1 step, while Frambo takes 5 steps forward and falls backwards by 4steps. Who will be the first to reach the (i) 100th step (ii) 123rd step?

Will they land on the same step at any point during the competition?

The Leapy Frog problem was adapted by Ms Wong Wai Mun and Ms Wong Wai See from: University of Mississippi "Ole Miss Problem of the week" http://www.olemiss.edu/mathed/brain/leap.html

Enhancing Mathematical Reasoning at Secondary School Level

Wong Khoon Yoong
National Institute of Education

This article is based on my keynote address at the Second Mathematics Teachers Conference held on 1 June 2006 at the National Institute of Education for the secondary school group. I hope that the ideas in this article are also relevant for primary school and Junior College teachers.

What is mathematical reasoning and why is it important

Reasoning by proof is the unique spirit of mathematics that differentiates it from other disciplines. The latest Singapore mathematics syllabus defines mathematical reasoning as "the ability to analyse mathematical situations and construct logical arguments" and notes that it is also "a habit of mind" (Ministry of Education, 2007, p. 5). The key idea is that logical arguments are used to convince oneself and others that a particular result is valid and can be defended against counter-arguments. When learning a new piece of mathematics, the students should be encouraged to ask questions like "Why is it true?" or "Why is it false?" so that they do not accept given results on faith. Mathematical reasoning becomes evident in problem solving when the students can state the formulae or theorems to support the intermediate steps. In both situations, patterns and regularities may be observed, and steps should be taken to check that these do not occur by accident. Mathematics teaching should aim to equip our students with this ability to reason logically and an inquiry mindset to do so not only in mathematics but also for many everyday situations. If the students fail to recall a learned rule, they should try to reason it out from first principles; just asking other people for the rule or looking it up in the textbook should be the last resort. This inquiry mindset is truly a hallmark of successful mathematics learners (school students as well as teachers).

Three local considerations have further highlighted the importance of reasoning. First, reasoning is one component of the *Processes* factor of the "Pentagon framework". Second, when Mr Tharman Shanmugaratnam, Minister of Education, declared open the *Philosophy in Schools Conference* in April 2006, he proposed that schools need to teach one more 'R' – reasoning – in addition to the "3 Rs" (Reading, wRiting, and aRithmetic). Third, the rubric for SAIL (Strategies for Active and Independent Learning) includes "Approach and Reasoning" as the first item in the assessment of students' investigation work. Thus, the students must be aware of the need to provide reasoning for their solutions. These considerations should convince teachers and students that mathematical reasoning is very important indeed.

True or False Statements and Definitions

The first thing students should learn about reasoning is that in mathematics a *true* statement applies to *every* relevant case covered by the statement. It is not acceptable to cite several (or even numerous) cases to justify a true statement. For example, consider the statement: For all whole numbers n, $n^2 + n + 41$ is a prime number. After checking that this holds for the first few cases (41, 43, 47, 53), a student may conclude that it is a true statement. At this stage, the crucial question to ask is: Is there any reason or "sufficient" evidence to support this conclusion other than those cases that have been checked? It is not easy to answer this question for complicated mathematics, and this is why there are still many unsolved problems in mathematics!

The second thing students should learn is that a mathematics statement is *false* if one can find just one example (called a counter-example) to show that it is false. The example above is false because the number $41^2 + 41 + 41$ is obviously not prime. Once a counter-example has been found, it is not necessary to find other counter-examples, although for pedagogical purpose, students may benefit from checking other counter-examples.

Definitions play a crucial role in reasoning. A pertinent question is: Can definitions be proved? Some teachers believe so, but actually definitions cannot be proved as true or false. They are generally agreed upon by a majority of mathematicians to be "useful" in describing a certain concept. The criteria for "usefulness" may include whether the definition is a fruitful extension of other concepts; whether it has strong physical justifications; or whether theorems about the concept will have simplicity and beauty. For example, we cannot prove that 1 is a prime number. Mathematicians have decided to exclude 1 as a prime number in order to preserve the *unique* prime factorization of natural numbers. If 1 were treated as a prime number, then composite numbers will not have unique prime factorization; for example, 6 = 2 x 3 = 1 x 2 x 3 = 1 x 1 x 2 x 3 and so on. This will spoil the beauty and simplicity of the Fundamental Theorem of Arithmetic. As another example, can we prove that $\pi = \frac{\text{circumference}}{\text{diameter}}$? Ancient people discovered from practical experience that $\frac{\text{circumference}}{\text{diameter}}$ is close to 3 for round objects, and this would have provided the rationale to give this fraction a special label, namely π. This does not *prove* that the label has to be π! Numerous mathematicians like Archimedes, Zu Chongzhi, and Wallis, try to obtain more and more accurate values of π through computations and various techniques. Teachers should help their students understand whether a given result is a definition or a property that can be derived from other properties by discussing many more examples, for examples: the probability of getting a Head when a fair coin is tossed is ½; sum of adjacent angles on a straight line is 180°; $180^\circ = \pi$ radians.

Intuitive-Experimental Justifications and Proofs

Two types of reasoning can be used to decide whether a statement is true or false: intuitive-experimental justifications and proofs. Intuitive-experimental justifications (IEJ) are based on practical activities with a variety of examples leading to some patterns or regularities, following the sequence: examples

→ conjecture → more examples → explanation or proof. Inductive thinking is involved here, and teachers should be aware of the danger of drawing the wrong conclusions from patterns and induction only. Thus, IEJ are not rigorous, but they are considered to be adequate for most students taking O-Level Mathematics. Teachers should try to provide IEJ for most of the results in the school syllabuses.

Mathematical proofs are more rigorous than IEJ and should be discussed after IEJ, as suggested by the van Hiele's Theory of geometry thinking, which is cited in the O-Level syllabuses. There are many methods of mathematical proofs, such as deduction, proof by contradiction, mathematical induction, proof by contrapositive, and so on. However, the O-Level Mathematics syllabus does not require these methods, but most students should be exposed to simple direct proofs. The Additional Mathematics syllabus requires elementary deductive proofs in plane geometry.

Rigorous proofs require precise definitions. Unfortunately, several mathematical terms in the school syllabuses do not have universally accepted, unambiguous definitions. For example, to decide whether an equilateral triangle is isosceles or not, one needs the definition of an isosceles triangle. According to Artmann (1999), Euclid defined an isosceles triangle as one with "two of its sides alone equal" (p. 18); with this definition, an equilateral triangle is *not* isosceles. On the other hand, many contemporary authors take the inclusive view and delete the "alone" part from the definition, in which case, an equilateral triangle *is* isosceles. Hence, there may be no universally agreed answer to the question about equilateral and isosceles triangles! In a similar vein, if a *trapezium* (or a *trapezoid* in American terminology!) is defined as a quadrilateral with one pair of parallel sides, does "one pair" mean the exclusive "only one pair" or the inclusive "at least one pair"? Everyday usage is exclusive, for example, when you say, two friends visited you, you mean exactly two. However, mathematical usage tends to be inclusive. Despite the perception that mathematics is a precise discipline, such ambiguities within mathematics and between mathematics and everyday language have caused much distress to teachers and students. This could be avoided, if when such tricky questions are asked, the intended definitions are also included. Then, it will be a test of logical thinking rather than an argument about definitions.

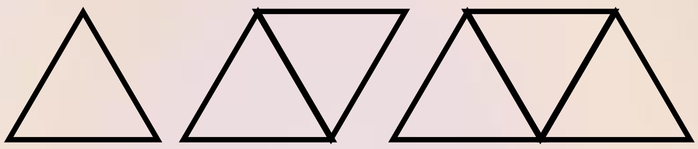
Given that imprecise definitions are confusing, it is in the interest of clarity of thought to train students to explore the implications of given definitions. Consider the following definitions taken from a local textbook: The general form of a quadratic polynomial is $ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$. This definition implies that x^2 is a quadratic polynomial since $a \neq 0$ although b and c are zero, a condition not required by the given definition. The authors also state that: A polynomial is an algebraic expression that contains more than two terms, especially the sum of terms containing different integral powers of the same variable(s). This implies that x^2 is *not* a polynomial since it has only one term. These conflicting deductions from the two given definitions provided by the same authors may not be even noticed by teachers who rush to cover (with pun) the contents, but the more discerning teachers should raise these tricky points at least with the more capable students. How would you deal with this definition: A quadratic expression in x is one in which the highest power of x is 2? It will be a useful exercise for teachers to compare the definitions given in all the approved textbooks and to discuss inconsistencies and ambiguities of definitions with their class.

At O-Level, there are, however, a few exceptions for which IEJ and proofs are too difficult for most students. Nevertheless, the students should be told that mathematicians have verified the results rigorously. Some examples are: $\sqrt{2}$ and π are irrational numbers; irrational numbers have non-terminating or non-recurring decimals; there are infinitely many prime numbers. Mathematics teachers should know the proofs themselves so that they can explain the gist of the proofs to those students who ask for justifications; these are, indeed, the future mathematicians that teachers need to nurture whenever the opportunities arise!

I will illustrate the link from IEJ to proof with three examples. In each case, a *variety* of different activities is the crucial consideration.

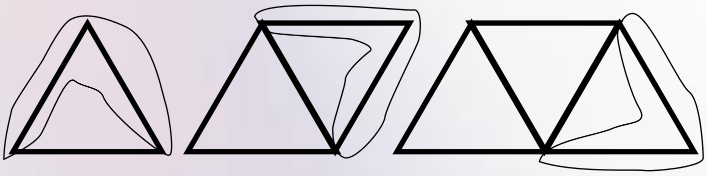
Example 1: Patterns Require Explanation

Ever since the Cambridge O-Level examination introduced "pattern questions" more than twenty years ago as a test of heuristic, these questions have even entered the primary classrooms. A typical example is to give the first three patterns, as illustrated below with arranging toothpicks into triangular patterns, and ask for the 100^{th} pattern (primary schools) or the algebraic formula (secondary schools).

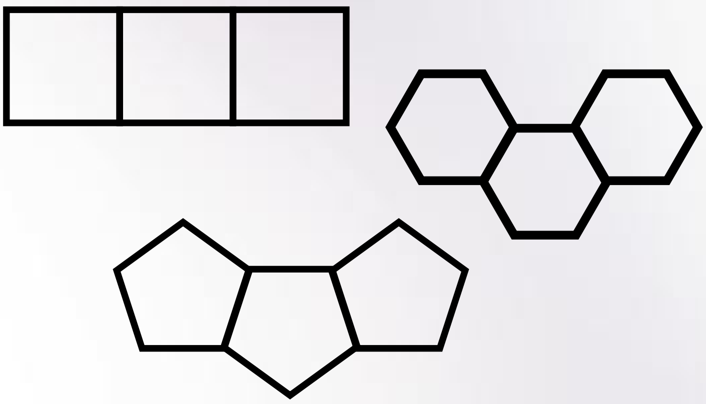


The expected approach is to make up a table of values, say 3, 5, 7, notice that these are odd numbers (the pattern), and then try to work out the 100^{th} odd number, *without referring again to the original, physical problem*. The italic part is crucial but most teachers hardly notice or discuss it. The implicit assumption is that the pattern from the first three cases will automatically extend in the "obvious" way (old numbers) and will tell us about the physical situation at the 100^{th} case.

An inquiry mind will articulate this assumption and challenge it. Must 9 be necessarily the next number? Since 3, 5, and 7 are the first three odd prime numbers, the pattern may continue as 11, 13, 17, and so on. Hence, it is necessary to give a justification why the physical arrangement will continue as odd numbers rather than as odd prime numbers. One way to do this is to notice that the next step is obtained from the preceding one by adding two more toothpicks, starting with one toothpick: 1 + 2; 3 + 2; 5 + 2, and so on. For primary school students, making the patterns with toothpicks will help them appreciate this process much clearer than through verbal explanation alone. This *physical arrangement* together with the pattern lead to the formula $1 + 2n$, with the physical arrangement providing the IEJ for the formula.



To help students "read" algebraic formula in a more concrete sense, we can rewrite $1 + 2n$ as $1 + n + n$ and ask which way of counting the toothpicks will lead to this new algebraic form; how about $(1 + n) + n$ or $2(n + 1) – 1$ or other equivalent forms? Repeat this exploration with patterns that begin as a square, a regular pentagon, a regular hexagon and so on.



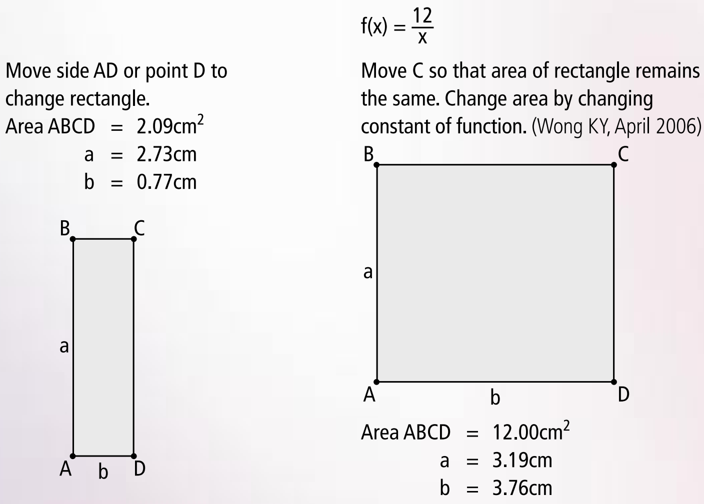
Can the students provide IEJ for the formula $1 + (k – 1)n$, where k is the number of sides of the first polygon? This "looking back" will help students develop their reasoning skills. See similar examples discussed by Lannin, Barker and Townsend (2006).

To further reinforce the need to link patterns to real arrangement, try this example. Take any two points on a circle. Join them with a chord. This divides the circle into two regions. Repeat this process with 3, 4, and 5 points. In each case, find the *greatest* number of regions formed inside the circle by the chords. The numbers of regions are: 2, 4, 8, and 16. Generalizing from this pattern, one would expect to get 32 regions with 6 points. Try this activity on a large piece of paper so that you can count the regions properly! Actually there are only 31 regions because the pattern follows a different rule: $1 + \binom{n}{2} + \binom{n}{4}$. The proof of this rule depends on understanding how each new chord adds to the number of regions (physical arrangement). The proof is difficult, but the important message is that IEJ with explanation (and proof, if possible) needs to be included when we ask students to solve these pattern questions. It is helpful to reiterate this point by citing Gardiner (1987): *Mere 'pattern-spotting' is not mathematics*!

Example 2: Zero Product Rule or Zero Factor Property

To solve quadratic equations by factorization, one needs to apply the zero product rule: If $ab = 0$, then $a = 0$ or $b = 0$. When asked how they would explain this rule to students, some teachers claim that it is too obvious to need an explanation, while others claim that this is true because any number multiplied by zero will give zero. The latter explanation is essentially about the converse of this rule: If $a = 0$, then $ab = 0$, for any b. To promote reasoning, one should not confuse an implication and its converse. A better justification for the zero product rule is required. Unfortunately, most textbooks just state the rule without justification.

Adopting an EFJ approach, we note that ab can be thought of as finding the *area* of a rectangle of sides a and b. Then the rule says that if the area of a rectangle is zero, then at least one of its sides must have zero length. This translation from symbolic statement to a practical situation makes the rule more sensible. Then let students explore it further in a "virtual" way: create a rectangle in Geometer's Sketchpad; let students change its area to zero and observe what happens to the sides; see (a) below. This exploration can be followed by a direct proof. Suppose $a \neq 0$. Then divide by a (division by non-zero numbers is permissible). We get $\frac{ab}{a} = \frac{0}{a}$, $b = 0$.



(a) (b)

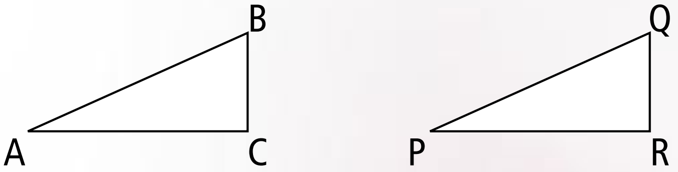
Most teachers would have come across the following over-generalization by their students: If $ab = c$, then $a = c$ or $b = c$. Just telling students that this new "rule" is false is not effective because it looks so convincingly obvious from the zero product rule. An IEJ approach is to begin with $ab = 12$, say, and ask the students to vary the rectangle but keeping its area to be 12 square units and to observe what happens to its sides; see (b) above. A direct proof will show the students that if $a \neq 0$, then $b = \frac{c}{a}$, which is not c unless $a = 1$. Hopefully, this combination of IEJ and proof will eliminate this misconception.

Example 3: Converse of Pythagoras Theorem

Unlike in the earlier versions of the O-Level syllabus, the term "converse" has been deleted from the current syllabus. Instead, it is stated: determining whether a triangle is right-angled given the lengths of the three sides. Some teachers cannot tell the difference between the Pythagoras Theorem and its converse. Getting rid of the "converse" label may not please those who believe that concept development should be followed by proper labels.

With reference to the diagram (c), Pythagoras Theorem states: if $C = 90^0$, then $c^2 = a^2 + b^2$.

The converse states: if $c^2 = a^2 + b^2$, then $C = 90^0$.



(c) (d)

There are already numerous proofs of the Pythagoras Theorem; for example, 72 proofs are given in http://www.cut-the-knot.org/pythagoras/index.shtml. However, the proof of the converse is less well known.

Begin with an IEJ activity. Write down two values *a* and *b*. Compute $c = \sqrt{a^2 + b^2}$. Use compasses to construct the triangle with sides *a*, *b* and *c*. Measure angle *C*. Is it close to 90^0? Repeat with different values. Then move to a virtual exploration with Geometers' Sketchpad: the user drags the vertex of a triangle until its sides satisfy the Pythagorean property; then note whether there is a right angle or not. See screen shot below.

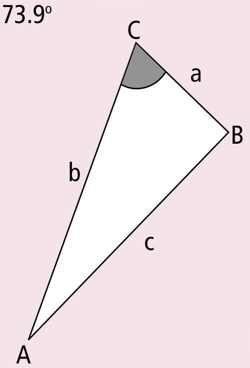
a = 1.497cm $\quad a^2 = 2.240cm^2 \quad$ Drag B so that $a^2 + b^2$ is equal to c^2.

b = 4.713cm $\quad b^2 = 22.214cm^2 \quad$ What happens to angle C?

c = 4.531cm

$a^2 + b^2 = 24.454cm^2$

Angle C = 73.9° $\quad c^2 = 20.534cm^2$



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Finally, consider this proof. Construct another triangle *PQR* (see (d) above) such that $QR = BC = a$, $PR = AC = b$, and $R = 90^0$. By Pythagoras Theorem, $PQ = \sqrt{QR^2 + PR^2} = c$. Hence, *PQR* is congruent to *ABC* (SSS). Hence, $C = R = 90^0$. On first encounter, some people may feel uncomfortable with this proof because it uses the Pythagoras Theorem to prove its converse. Clear understanding is important, but familiarity is an added advantage. An alternative proof is to apply the Cosine's Rule.

Concluding Remarks

We should expect *all* students to engage in mathematical reasoning beginning at a level that is appropriate for their mathematical sophistication. Encourage them to talk about their reasoning, to listen to the justifications given by their classmates, to ask questions about reasons, to explore alternatives, and to write down their reflection of such interactions. Many students look toward the teacher as models, so teachers need to demonstrate in front of the class their own reasoning and the disposition to ask for justifications and proof. The teacher's ultimate goal is to help students to appreciate different ways of mathematical reasoning, preferably building on IEJ. Some interesting articles for further reading are given in the list below.

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The two GSP files are available from my website at: http://math.nie.edu.sg/kywong/