ASSOCIATION OF MATHEMATICS EDUCATORS | http://math.nie.edu.sg/ame/

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President's Message

Dear AME members,

The new mathematics curriculum puts its emphasis on learning experience. It is important to note that how students learn mathematics is at least as important as what they learn in mathematics classrooms. This is an important message to all mathematics teachers and educators. Thus we have chosen AME activities to help teachers grapple with learning experience in mathematics.

Last year, AME jointly organized the AME-SMS Conference 2013 with the Singapore Mathematical Society. The theme of the conference was "Learning Experience in Mathematics". We are encouraged that 655 teachers from Singapore and abroad attended this event. We have continued to choose a theme that is of immediate relevance to mathematics teachers and, also, one that is broad enough to engage both mathematics educators and mathematicians to come together and offer valuable knowledge of content and pedagogy to the participating teachers.

For teachers who wanted to have intensive workshop to acquire the knowledge and skills to design appropriate learning experiences in their mathematics lessons, AME continued to organize her second AME institute for primary and secondary school teachers. Here we would like to put on record our appreciation for the instructors Ms Juliana Ng and Ms Liu Yueh Mei for jointly conducting the 10-hour session for primary school teachers on 18-19 March 2013, Dr Hang Kim Hoo and Dr Lee Chan Lye for conducting the 10-hour session for secondary school teachers on learning experience in Geometry on 4-5 June 2013. We would also like to thank our foreign delegates Professor Margaret Brown and Professor Jeremy Hodgen for conducting another 10-hour session for secondary school teachers on learning experience in Secondary School Algebra.

AME will continue to reach out to mathematics teachers through her activities. The committee members always take great care to ensure that the activities that AME organizes will be relevant and timely to school teachers.

I would like to take this opportunity to thank you for your continued support of AME. In the meantime, enjoy reading this issue of Mathbuzz.

Toh Tin Lam
President,
AME (2012 – 2014)

2013 Events Organized by AME

Near capacity crowd at NUS High School Auditorium for the AME-SMS Conference 2013.

Participants of the AME-SMS Conference 2013.

Contents

Editorial Committee Members:
Dr Toh Pee Choon & Dr Lee Ngan Hoe

TIMSS 2011

- Performance in mathematics of eighth graders from Singapore and the Asia Pacific countries

Berinderjeet Kaur - National Institute of Education

What is TIMSS 2011?

TIMSS (Trends in International Mathematics and Science Study) 2011 is the fifth in a series of international mathematics and science assessments conducted periodically every four years. TIMSS is designed to provide trends in fourth- and eighth- grade mathematics and science achievement in an international context. The aim of TIMSS is to provide policy makers with a wealth of information about key instructional, curricular, and resource related variables that are fundamental in understanding the teaching and learning process. In TIMSS 2011, 45 countries participated at the eighth grade level. The Asia-Pacific countries that participated at the eighth grade were Australia, Chinese Taipei, Hong Kong, Indonesia, Japan, Korea, Malaysia, New Zealand, Singapore and Thailand. Data was collected from participating students, their teachers and school leaders with the help of assessment tasks and background questionnaires. The TIMSS 2011 International Results in Mathematics (Mullis, Martin, Foy & Arora, 2012) is a comprehensive report of all the data collected and analysed for mathematics assessment of grades four and eight students. This article draws on the data from the report and reports on the achievement of grade eight students from Singapore and the Asia Pacific countries that participated in the study.

Student Participants and Tests

Representative samples of eighth graders participated in the study. They were in their eighth year of formal schooling with average ages ranging from 14.0 to 14.5 years. The TIMSS 2011 tests comprised of both mathematics and science items. Fourteen different booklets containing a selection of the 215 mathematics and 217 science items were administered to the students. Each student completed the test in one booklet. Testing time was 90 minutes. The 217 mathematics items (118 multiple choice and 99 constructed response type) were classified by content domain and cognitive domain. The four content domains were Number, Algebra, Geometry, and Data and Chance, while the three cognitive domains were Knowing, Applying and Reasoning (Mullis, Martin, Ruddock, Sullivan & Preuschoff, 2009).

Mathematics Achievement

Table 1: Rank and average scale scores of Asia-Pacific countries

	TIMSS 2011		TIMSS 2007	
Country	**Rank**	**Average Scale Score**	**Rank**	**Average Scale Score**
Korea, Rep of	1	613 (2.9)	2	597 (2.7)*
Singapore	2	611 (3.8)	3	593 (3.8)*
Chinese Taipei	3	609 (3.2)	1	598 (4.5)*
Hong Kong, SAR	4	586 (3.8)	4	572 (5.8)
Japan	5	570 (2.6)	5	570 (2.4)
Australia	12	505 (5.1)	14	496 (3.9)
International Avg	-	500		
New Zealand	16	488 (5.5)	-	-
Malaysia	26	440 (5.4)	20	474 (5.0)
Thailand	28	427 (4.3)	29	441 (5.0)
Indonesia	38	386 (4.3)	36	397 (3.8)

Standard errors are shown with ( ).
* No significant difference between average scale scores

Table 1 shows the ranking and average scale scores of the Asia-Pacific countries that participated in TIMSS 2011 and TIMSS 2007. The five East Asian countries were in the top five ranks for both TIMSS 2011 and TIMSS 2007. Korea, Singapore, Chinese Taipei, Hong Kong and Australia improved their average scale scores in 2011 compared to 2007, and Australia made a significant upward move to the 12th place with an average scale score higher than the international average of 500.

International Benchmarks of Mathematics Achievement

The international benchmarks presented as part of the TIMSS 2011 data (Mullis, Martin, Foy & Arora, 2012) help to provide participating countries with a distribution of the performance of their eighth-graders in an international setting. For a country the proportions of students reaching these benchmarks perhaps describe certain strengths and weaknesses of mathematics education programs of the country. The benchmarks delineate performance at four points of the performance scale. Characteristics of students at each of these four points are elaborated in the next section.

Table 2 shows the percentage of students from the Asia-Pacific countries reaching TIMSS 2011 international benchmarks of mathematics achievement. It is worthy to note that almost half of the students from Chinese Taipei, Singapore and Korea were at the Advanced benchmark. Furthermore in all the five Asia-Pacific countries that were ranked as the top five, more than 70% of their students were at the High benchmark level, with the exception of Japan (61%) and almost 90% of their students were at the Intermediate benchmark level. In contrast, particularly for Australia, only 9% of their students were at the Advanced Benchmark level and 29% at the High benchmark level. Nevertheless the proportions of students from Australia at the advanced and high benchmarks have improved compared to TIMSS 2007. In TIMSS 2007, only 6 % were at the advanced level and 24 % were at the high level. For New Zealand, only 5% were at the advanced level and 16% of the students were below the low level. For Malaysia, Thailand and Indonesia, only 2% or less of the students were at the advanced level and 35%, 38% and 57% of students respectively were below the low level.

Table 2: Percentages of students reaching TIMSS 2011 international benchmarks of mathematics achievement.

Country	Advanced benchmark (625)	High benchmark (550)	Intermediate benchmark (475)	Low benchmark (400)
Chinese Taipei	49 (1.5)	73 (1.0)	88 (0.7)	96 (0.4)
Singapore	48 (2.0)	78 (1.8)	92 (1.1)	99 (0.3)
Korea, Rep of	47 (1.6)	77 (0.9)	93 (0.6)	99 (0.2)
Hong Kong SAR	34 (2.0)	71 (1.7)	89 (0.7)	97 (0.3)
Japan	27 (1.3)	61 (1.3)	87 (0.7)	97 (0.3)
Australia	9 (1.7)	29 (2.6)	63 (2.4)	89 (1.1)
New Zealand	5 (0.8)	24 (2.6)	57 (2.8)	84 (1.6)
Malaysia	2 (0.4)	12 (1.5)	36 (2.4)	65 (2.5)
Thailand	2 (0.4)	8 (1.3)	28 (1.9)	62 (2.1)
Indonesia	0 (0.1)	2 (0.5)	15 (1.2)	43 (2.1)
International Median	2	17	46	75

Standard errors are shown with ( )

What can students at each of these benchmarks do?

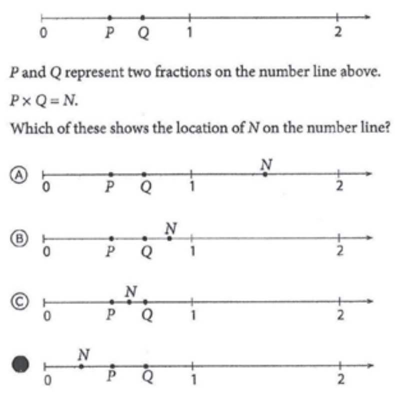
Advanced International benchmark

At the advanced international benchmark students can

- reason with information, draw conclusions, make generalizations, and solve linear equations;
- solve a variety of fraction, proportion, and percent problems and justify their conclusions;
- express generalisations algebraically and model situations;
- solve a variety of problems involving equations, formulas, and functions;
- reason with geometric figures to solve problems; and
- reason with data from several sources or unfamiliar representations to solve multi-step problems.

Figure 1 shows an item that students reaching the advanced benchmark were likely to answer correctly.

Figure 1. An advanced international benchmark item.

Content Domain: Number Cognitive Domain: Reasoning Description: Given two points on a number line representing unspecified fractions, identifies the point that represents their product	Country	Percent correct
	Chinese Taipei	53 (2.0)
	Hong Kong SAR	47 (2.5)
	Singapore	45 (2.0)
	Korea, Rep of	44 (2.0)
	Japan	43 (2.1)
	Australia	23 (2.1)
	New Zealand	19 (2.3)
	Malaysia	18 (1.4)
	Thailand	12 (1.5)
	Indonesia	10 (1.7)
	International Avg	23 (0.3)

Standard errors are shown with ( ).

High International benchmark

At the high international benchmark students can

- apply their understanding and knowledge in a variety of relatively complex situations;
- use information from several sources to solve problems involving different types of numbers and operations;
- relate fractions, decimals, and percents to each other;
- basic procedural knowledge related to algebraic expressions;
- use properties of lines, angles, triangles, rectangles, and rectangular prisms to solve problems; and
- analyse data in a variety of graphs.

Figure 2 Shows an item that students reaching the high benchmark were likely to answer correctly.

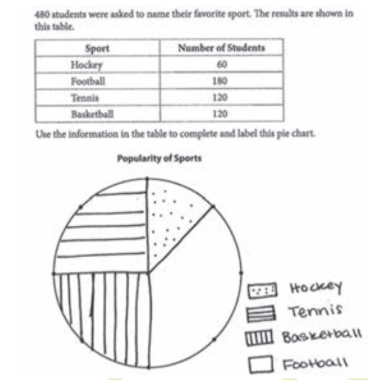
Intermediate International benchmark

At the intermediate international benchmark students can

- apply basic mathematical knowledge in a variety of situations;
- solve problems involving decimals, fractions, proportions, and percentages;
- understand simple algebraic relationships;
- relate a two-dimensional drawing to a three-dimensional object;
- read, interpret, and construct graphs and tables; and
- recognise basic notions of likelihood.

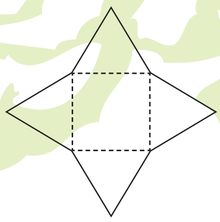
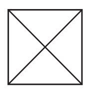
Figure 3 shows an item that students reaching the intermediate benchmark were likely to answer correctly.

Figure 2. A high international benchmark item.

Content Domain: Data and Chance Cognitive Domain: Applying Description: Constructs and labels a pie chart representing a given situation	Country	Percent correct
	Singapore	85 (1.5)
	Korea, Rep of	85 (1.4)
	Chinese Taipei	80 (1.7)
	Hong Kong SAR	76 (1.8)
	Japan	75 (1.7)
	Australia	67 (2.3)
	New Zealand	59 (2.5)
	Malaysia	50 (2.2)
	Thailand	45 (2.3)
	Indonesia	28 (2.2)
The answer shown illustrates the type of student response that was given 2 of 2 points	International Avg	47 (0.3)

Standard errors are shown with ( ).

Figure 3. An intermediate international benchmark item.

Content Domain: Geometry Cognitive Domain: Knowing Description: Given a net of a three-dimensional object, completes a two-dimensional drawing of it form a specific viewpoint	Country	Percent correct
	Japan	89 (1.2)
	Australia	87 (1.2)
	Korea, Rep of	85 (1.3)
	New Zealand	84 (1.7)
	Singapore	83 (1.4)
	Hong Kong SAR	77 (2.0)
The shape shown above is cut out of cardboard. The triangle flaps are then folded up along the dotted lines until they touch the edges of the flaps next to them.	Chinese Taipei	74 (1.7)
Complete the diagram below to show what the shape would look like when viewed from directly above.	Malaysia	53 (1.8)
	Thailand	51 (2.4)
	Indonesia	27 (2.2)
 The answer shown illustrates the type of student response that was given 1 of 1 points	International Avg	58 (0.3)

Standard errors are shown with ( ).

Low International benchmark

At the low international benchmark students have some knowledge of whole numbers and decimals, operations, and basic graphs. Figure 4 shows an item that students reaching the low benchmark were likely to answer correctly.

Figure 4. A low international benchmark item.

Content Domain: Algebra Cognitive Domain: Knowing Description: Evaluates a simple algebraic expression	Country	Percent correct
$y = \frac{a+b}{c}$ $a = 8$, $b = 6$, and $c = 2$ What is the value of y? ● 7 Ⓑ 10 Ⓒ 11 Ⓓ 14	Korea, Rep of	92 (1.0)
	Chinese Taipei	91 (1.0)
	Singapore	91 (1.1)
	Japan	86 (1.5)
	Hong Kong SAR	83 (1.8)
	Australia	71 (2.6)
	Indonesia	65 (2.4)
	New Zealand	61 (2.6)
	Thailand	56 (2.2)
	Malaysia	47 (2.1)
	International Avg	71 (0.3)

Standard errors are shown with ( ).

Some Difficult Items

Figures 5, 6, 7, and 8 show four items that relatively more secondary two students from Singapore found difficult.

Figure 5. A relatively difficult item for Singapore students – Item 1

Content Domain: Number Cognitive Domain: Applying	Country	Percent correct
Kim is packing eggs into boxes. Each box can hold 6 eggs. She has 94 eggs. What is the smallest number of boxes she needs to pack all the eggs? Answer: 16 boxes	Hong Kong SAR	89 (1.3)
	Chinese Taipei	86 (1.4)
	Japan	75 (1.9)
	Korea, Rep of	74 (1.9)
	Singapore	64 (2.0)
	Australia	53 (2.9)
	New Zealand	46 (2.1)
	Thailand	33 (2.2)
	Malaysia	29 (1.7)
	Indonesia	21 (2.1)
	International Avg	41 (0.3)

Figure 6. A relatively difficult item for Singapore students – Item 2

Content Domain: Algebra Cognitive Domain: Applying	Country	Percent correct
What is the sum of the three consecutive whole numbers with 2n as the middle number? 6n + 3 6n 6n – 1 6n – 3 Answer: B	Korea, Rep of	78 (1.8)
	Chinese Taipei	76 (1.9)
	Hong Kong SAR	69 (2.0)
	Japan	68 (1.7)
	Singapore	64 (1.7)
	Thailand	52 (1.8)
	Indonesia	46 (2.0)
	Australia	45 (2.2)
	Malaysia	45 (2.0)
	New Zealand	43 (2.3)
	International Avg	52 (0.3)

Figure 7. A relatively difficult item for Singapore students – Item 3

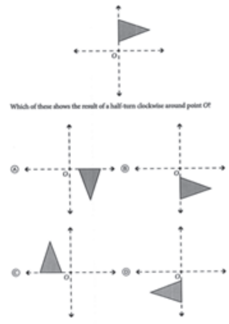
Content Domain: Algebra Cognitive Domain: Knowing Description: Evaluates a simple algebraic expression	Country	Percent correct
 Answer: D	Korea, Rep of	79 (1.7)
	Japan	73 (1.8)
	Hong Kong SAR	71 (2.0)
	Chinese Taipei	64 (1.6)
	New Zealand	63 (2.2)
	Singapore	62 (1.6)
	Australia	55 (2.6)
	Thailand	44 (2.0)
	Malaysia	44 (1.9)
	Indonesia	40 (2.0)
	International Avg	45 (0.3)

Figure 8. A relatively difficult item for Singapore students – Item 4

Content Domain: Data and Chance Cognitive Domain: Knowing	Country	Percent correct
Pat and Chris were candidates for school president. Here are the election results: Pat 80% Chris 20% How likely would it be for a student asked at random to have voted for Pat? It is certain that the student voted for Pat. It is likely that the student voted for Pat. It is unlikely that the student voted for Pat. It is certain that the student did not vote for Pat. Answer: B	Korea, Rep of	95 (0.8)
	Chinese Taipei	89 (1.3)
	Japan	88 (1.3)
	Hong Kong SAR	87 (1.7)
	Australia	84 (1.7)
	New Zealand	80 (1.9)
	Singapore	77 (1.4)
	Thailand	52 (2.1)
	Malaysia	37 (1.9)
	Indonesia	35 (2.1)
	International Avg	64 (0.3)

Where you can get more items and data?

After every cycle of the TIMSS, approximately half of the test items are released for public use. For TIMSS 2011 you may view the released items with useful data at the following webpages:

- http://www.nie.edu.sg/centre-international-comparative-studies/trends-international-mathematics-and-science-study-timss-2011
- http://timss.bc.edu/timss 2011

A study of the released items and the data for our students may shed light on some of our classroom practices and the cognitive aspects of tasks that we often use for our instructional needs.

Acknowledgement

The source of the items used in this article is: TIMSS 2011 Assessment. Copyright © 2013 International Association for the Evaluation of Educational Achievement (IEA). Publisher: TIMSS & PIRLS International Study Centre, Lynch School of Education, Boston College, Chestnut Hill, MA and IEA, IEA Secretariat, Amsterdam, the Netherlands.

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My Views on Primary School Mathematics

Pearlyn Gan - National Institute of Education

I had the privilege of heading the mathematics department in a primary school from 2005 to 2007, then from 2010 to 2011 before being seconded to NIE as a teaching fellow in 2012. In the earlier years, the department's focus was on making mathematics fun in school. Our department came up with a handbook of interesting mathematics activities and set up games stations with the help of parents during recess to instill the love for mathematics in our pupils.

In 2010 and 2011, our focus changed as we were in dire need of improving our PSLE results. Our percentage of quality passes had plunged by about 8% in 2009 and my colleagues and I resorted to firefighting. We came up with worksheets following the style of "successful" assessment books – we categorized problems, wrote examples for each type of problems and had practices where pupils would solve the problems according to our examples.

Our quality passes did inch up slightly, but I am not sure if that could be attributed to the exam-oriented style of teaching. I personally used the materials we produced with zest, believing that we had created the perfect PSLE manual. From my observations, the handful of mathematics whizzes in my class would have aced their exams regardless of what methods we used. The middle achievers could cope with the easier types of problems, while the low achievers struggled with the whole manual. I realized that in our worksheets, we had strayed away from the model method, focusing instead on recently coined methods such as branching. My conclusion is that low and middle achievers needed to visualize the problems, so doing away with drawing of diagrams was not a good idea. Furthermore, although useful for exams, categorizing of problems did not aid in conceptual understanding.

The experience propelled me to think about how I would teach when I return to a primary school after my secondment. I share my thoughts below:

Introducing Concepts

1) I am grateful to CPDD for providing the plethora of resources and I will certainly use the CPA (Concrete-Pictorial-Abstract) approach to deliver concepts. I will conscientiously make the links between the three modes of representation as I have seen how the failure to do so can result in pupils' confusion.

2) I have not tried and tested the point I am about to make, but I think it is worth experimenting. The idea was shared by an award-winning teacher from China. The rationale of her approach was to encourage pupils' self-directed learning. The teacher would allocate time during her mathematics lessons for pupils to read their textbooks. She would then ask them what new information they had gathered from reading. The pupils discuss questions raised and hopefully, misconceptions would be corrected at this stage. Her belief is that teachers do not need to teach what pupils can acquire on their own.

3) I used to make it a point to conduct at least one hands-on activity for each topic and I was happy to see my pupils engaged during such activities. Using M&M's to discuss ratio, cutting fruits or cakes when learning about what a fraction of a fraction or percentage means, finding the best angle to shoot a bow in an online archery game were some of the activities my colleagues and I used. We should always be mindful of the mathematical objectives of our lessons though and not let the fun mar what still have to be delivered.

Teaching Skills

4) From my observation, most teachers are comfortable teaching mathematical skills. To make it easier for middle and low achievers to remember procedures, we can break them down into smaller steps. For instance, for addition of 4 digit numbers with renaming, we can say:

   Step 1: Start from the ones place. Add the 2 digits.

   Step 2: Do you need to regroup? If not, go on to the tens place. If yes, regroup the ones to tens and ones.

   And so on.

5) One useful thing I learnt here in NIE is that while planning the consolidation phase of a lesson, we should always ask ourselves if our pupils are prepared to do their homework. I find that we often assign homework that is much more difficult than what we have taught (especially in upper primary) and pupils return home only to struggle through the assignment. The consolidation phase should consist of teacher modeling, guided practice and independent practice. In teacher modeling, the teacher models and articulates the steps. In guided practice, the teacher asks pupils questions such as "What is the first step?" or "What do you do next?" etc. and expects pupils to articulate the steps that have been demonstrated before. Independent practice is as its name suggests.

Problem Solving

6) For problem-solving, one apparent and age-old strategy is to break down the information into parts. I will ask my pupils to read the problem as a whole first, then sentence by sentence. While reading sentence by sentence, they can translate English into mathematics and make annotations beside or below the problem. This is to ensure that pupils understand the problem. It is also to minimise the amount of information that pupils have to process.

   Below is a PSLE problem with my annotations:

   In a school hall, chairs were arranged in rows such that there were exactly 9 chairs in each row.

   For a concert, Mr Ong brought 6 more chairs into the school hall and rearranged all the chairs. There are now exactly 7 chairs in each row and 12 more rows than before.

   How many chairs are there in the school hall for the concert?

   Possible annotations:
   Before: 1 row - 9 chairs
   After: + 6 chairs
   1 row - 7 chairs
   12 more rows than before
   Goal: Number of chairs after

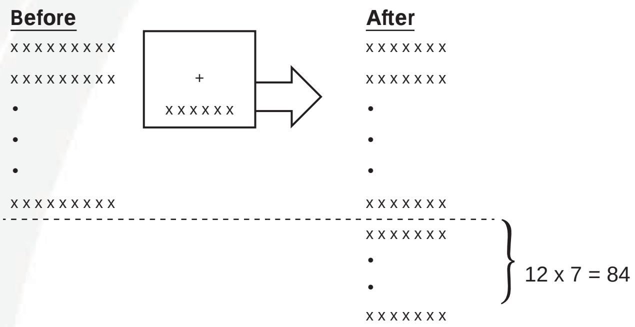
6) Instead of categorizing problems into types for pupils, I will teach them useful heuristics and encourage them to find out which heuristics are most appropriate for each problem. I personally find *"Drawing a Diagram"* most useful for solving primary school problems. Most PSLE problems could be solved or better visualized when diagrams are drawn.

   Making a List and Guess and Check could be tedious and I know the exam authorities discourage the use of the latter. However, these are useful last resorts and I will still show pupils how to use them for varied kinds of problems. Having said that, I will stress that these should not be the pupils' immediate methods of choice.

Working Backwards, Making a Supposition and Looking for Patterns are pertinent to specific kinds of problems and should also be taught.

Using the same problem in (6), I will show how *Drawing a Diagram and Guess* and Check could be used to solve the problem.

Drawing a Diagram



The 12 rows of 7 chairs are made up of the extra 6 chairs as well as the rows of 2 in the Before state.

Thus, the first step is to subtract 6 from 84, then divide by 2 to find the number of rows of extra twos.

Solution:

Before 6 chairs were added : 84 – 6 = 78
No. of rows of extra twos : 78 ÷ 2 = 39 (equal to the number of rows of 9)
No. of rows of 7 : 39 + 12 = 51
No. of chairs at the concert : 51 x 7 = 357

Guess and Check

No. of rows at first	No. of chairs at first (No. of rows at first x 9)	No. of chairs + 6	No. of rows of 7	Is no. of rows of 7 = 12 + no of rows of 9?
10	90	96	96÷7≈13.7	No
20	180	186	186÷7≈26.6	No
30	270	276	276÷7≈39.4	No
40	360	366	366÷7≈52.3	No
39	351	357	357÷7=51	Yes

From the guesses in lines 1-4, it can be seen that the difference between Columns 1 and 4 increases. In line 4, the difference is about 12.3. Since we want a difference of 12 rows, the original number of rows should be between 30 and 40. Hence, in line 5, I reduced the number of rows by 1 and got the answer.

I will highlight the tedium of the Guess and Check method while demonstrating it to pupils. Alternatively, pupils could be asked to compare the two methods and decide which is more appropriate. Hopefully, they will not have to resort to using the Guess and Check method during an exam. Also, many pupils guess and fail to check. They have to be reminded to check all conditions if they are using this method.

8) I will also show my pupils how to check their answers by working backwards, using another strategy or ensuring reasonableness. Normally, the high achievers are the ones who have time to do this during an examination. Still, I will equip every pupil with such skills as accuracy is important in mathematics.

For the same problem on chairs, if pupils had used the first method to solve the problem, they could use another approach to check their answers.

Instead of using 51 x 7, they could use 39 x 9 to find out the original number of chairs, then add 6 to check if they obtain 357.

9) My former school worked with a group of mathematics level heads, who came up with the acronym STAR to help pupils remember Polya's 4-step problem solving model. STAR stands for See, Think, Act and Relook. I think helping pupils to be conscious of these 4 steps is equivalent to equipping them to be independent problem solvers.

General

10) For mixed ability classes, open-ended questions could be used. These questions are useful as students who work faster could be encouraged to come up with more solutions. The slower ones could be asked to produce just one or two solutions within a given time frame. Everyone gets his practice and the high achievers do not waste time waiting for others. An example of an open-ended question could be "In a family of five, their average age is 15. What are the possible ages of the 5 family members?"

11) I will continue to think of ways to help pupils connect mathematics to real life. Here are some examples:

a. One of my student-teachers brought a huge packet of Milo satchets that said "15% more" when she was demonstrating what an increase of 15% meant. We explored claims of "___% more" on other products and found that such claims were not always true.

b. When introducing speed in Primary 6, we could get pupils to discuss speeds of other vehicles like bullet trains, or speeds of animals and get pupils to arrange animals in order of their running speeds.

12) Other ways to make mathematics lessons interesting include the use of stories, newspapers, investigative tasks, magic tricks, videos, the history of mathematics and songs. Recently, a student-teacher wrote an excellent story called The Tragic Tale of the Lines Which Will Never Meet. It is not difficult to guess what topic she was teaching. Another student-teacher transformed himself into the owner of a provision shop in a video on money.

With the advancement of technology, there is no limit to creativity. You could let your pupils have a good laugh over your photo superimposed on dollar bills in a lesson on money.

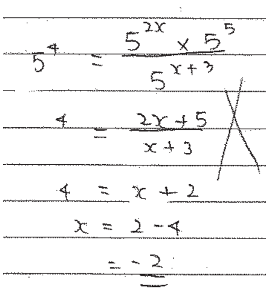
These are my humble thoughts on primary mathematics. Mathematics is exciting and I hope that we will all transmit this message to our pupils with our passion.

An Investigation of Students' Errors and Misconceptions in Exponents

Rosalind Lee - Anderson Secondary School

In the Singapore curriculum, students have been taught to learn exponential functions as formulas associated with the rules using routine algorithms and students are drilled in their application of laws of indices as a means of coping with exponential functions. Little attention is devoted to the conceptual development of exponents. Therefore, students often encounter difficulties dealing with exponents. Errors and misconceptions are evident in their homework and tests (Figure 1) due to a lack of understanding of letters and symbols, structures of expressions and equations, such as, $4(2^x) = 8^x$ and the procedural, and structural understanding of exponents. The errors are more prevalent in expressions involving negative or fractional powers such as $(9)^{-\frac{3}{2}} = {}^{-2}\sqrt{9^3}$. Exponents with negative or rational indices were primarily the cause of misconceptions and $a^{x+y} = a^x + a^y$ and $a^{xy} = a^x a^y$ were common errors made because of overgeneralization of the distributive property $a(x+y) = ax + ay$

Figure 1. Sample of a student's solution, surfacing errors made in a common test.



The data reported here come from a study carried out on Secondary Three Normal Academic students that investigated their errors and misconceptions in working with exponents.

Errors involving the base

The most common errors made were items with negative indices, $(-4)^{-1}$, where the base is a negative integer and students' displayed surface understanding of the number system between fractions and negative values, evident in errors such as $-4 = \frac{1}{4}$ (Figure 2).

Figure 2. Evidence of students' lack of understanding of the base.

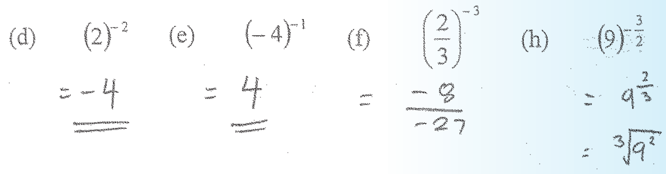
Errors involving the base and repeated multiplication

Students could associate a^3 with $a \times a \times a$ as it contains a single term but failed to transfer this procedural knowledge to $(-2b)^2$ and $(x^{\frac{3}{4}})^2$. Some students did not connect perfect squares to repeated multiplication probably because $(-2b)^2$ contains a negative sign and is composed of the product of -2 and b whereas $x^{\frac{3}{4}}$ has a fractional index. Although some students were able to connect their procedural knowledge of repeated multiplication with negative bases, they displayed an instrumental understanding of multiplication.

Errors involving exponents with negative indices

Most of the average and low ability students were handicapped in their understanding of negative indices and their rules. They made errors such as those shown in Figure 3, where in some cases they multiplied the negative power to the base. Other students simply took the reciprocal of the index instead of the base because of their incorrect understanding of "taking reciprocal".

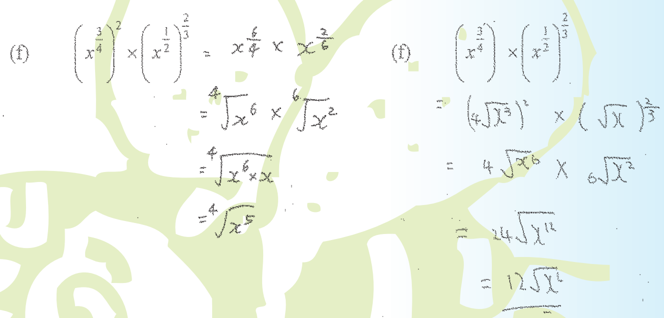
Figure 3. Sample of students' errors involving exponents with negative index.



Errors involving exponents with fractional indices

Most students possessed structural knowledge in fractional indices when the index of the exponent was $\frac{1}{n}$. Errors were more prevalent amongst the average and low ability students when the index was $\frac{m}{n}$. Figure 4 shows a gap in students' understanding of bi-directional connection between radicals and exponents, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$. Students could not resonate with the fact that radicals can be expressed in index form and the rules of exponents also apply. This is attributed to their instrumental understanding of the meaning of radicals.

Figure 4. Errors involving radicals and exponents.



Errors due to a lack of structural knowledge

The structure in expressions inhibited the understanding of exponents in students as they failed to understand the syntax of the expressions (Figure 5). Generally, students failed to recognize the rule $(ab)^m = a^m b^m$ to apply it in $(4x^6)^{\frac{1}{2}}$ and they simply "removed" the brackets, $(4x^6)^{\frac{1}{2}} = 4x^{-\frac{2}{6}}$, a common error due to lack of structural understanding and this error was transferred to $2(10^x) = 20^x$ and $\frac{1}{2}(4^x) = 2^x$. Hence, students showed that they could not accept a lack of closure in expressions. Another common error made was the omission of the brackets (Figure 6).

Most students depended on their prototypes such as $\frac{1}{2}(4x)$ and applied the same procedure to $\frac{1}{2}(4^x)$. These students revealed that they did not understand the semantics of x in $4x$ and 4^x and did not give conceptual meaning to the two terms. Hence, $4x$ and 4^x were treated as the same. When the expressions showed greater structural complexity, students simply co mbine the elements (Figure 7) without giving meaning to the expressions.

Figure 5. Sample of students' errors due to poor understanding of structure.

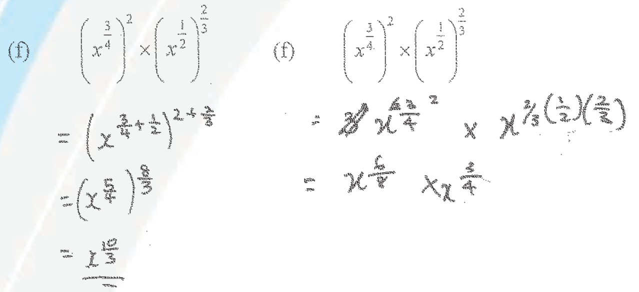


Figure 6. Omission of brackets in index.

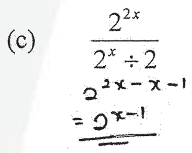
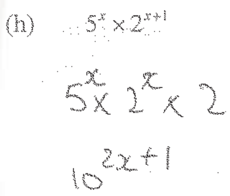


Figure 7. Student combining elements in expression.



Procedural knowledge.

The false generalization of "cancelling" of bases (Figure 8), formed by modifying their prior knowledge in "cancelling" common factors in algebraic fractions was a common error. Another prevailing error is due to overgeneralization where some students treated the base as a common factor (Figure 9). These are evidence of errors of interference as students were accustomed to their procedures in algebraic manipulations and incorrectly applied them to exponents.

Figure 8. False generalization of "cancelling" of bases.

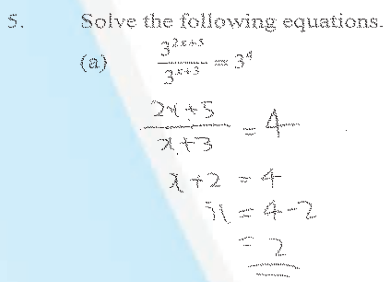
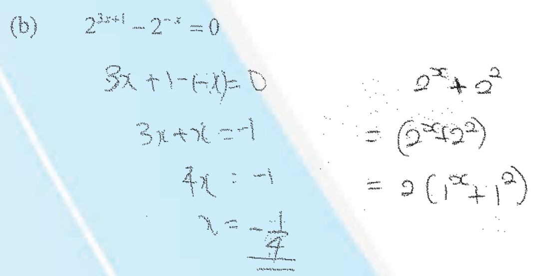


Figure 9. Examples of overgeneralization.



The research highlighted that students had a good grounding in understanding exponents with integer powers, even in items that contained multi-terms. However, students' weaknesses lie primarily in negative and fractional indices. The low ability students displayed surface understanding of the number system which caused errors in exponents with negative and fractional bases. The lack of closure and syntax of structures hindered students' understanding of the meaning of expressions. Students also disclosed that they did not understand the semantics of exponents and their inadequate procedural knowledge resulted in overgeneralization of the distributive property or the false generalization of "cancelling" of terms in exponents.

The essence of these errors and misconceptions, resulting in students' learning difficulties is due to their surface understanding in procedural, structural, and conceptual knowledge of exponents and the rules. These are attributed by students' prior knowledge, prototypes, and their surface understanding of written symbols. Hence, many errors and misconceptions arose in the learning of exponents.

Implications on Teaching and Learning of Exponents

There should be concerted effort by teachers to relate exponents to repeated multiplication and integrate the definition of exponent ax as "the number that is the product of x factors of a" to reinforce structural understanding, which aids understanding of rules of exponents. Students should also be provided with more experiences in the interpretation of exponential forms. Teachers can illuminate the meaning to procedures and rules by making connections to the basic properties of the number system and explain why the rules of exponents work. Though mastery of syntactic and algebraic rules strengthens the structural forms in exponents, it is essential for teachers to assign meaning to abstract structures when teaching exponents.

Error analysis provides a platform for teachers to gain insights into students' mental constructions and development of meaningful understanding of exponents. Teachers may highlight the common errors made in exponents or using errors as non-examples to address misconceptions during teaching and to engage students in discussions of these errors. Worksheets with both the correct and incorrect solutions are springboards for cognitive conflict to provoke students' thinking through error analysis.

Suggested Readings

- Farah-Sarkis, F. (1993). Sources of misconceptions: The case of powers and radicals. In the Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics. Retrieved February 3, 2011 from http://www.mlrg.org/proc3pdfs/Farah-Sarkis_Powers.pdf
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- Pitta-Pantazi, D., Christou, C., & Zachariades, T. (2007). Secondary school students' levels of understanding in computing exponents. Journal of Mathematical Behaviour, 26, 301-311.
- Weber, K. (2002a). Developing Students' Understanding of Exponents and Logarithms. In Proceedings of the 24th Annual Meeting of North American Chapter of Mathematics Education (Vols. 1-4). Retrieved February 4, 2011 from http://www.eric.ed.gov/PDFS/ED471763.pdf

Reflection of y=f(x) about a line y=kx

Kwee Tiow Choo - Hwa Chong Institution

While attending a conference, a mathematics teacher told me that he posted a question to his students, and no one has solved the question.

Question: Given a function $y = f(x)$, find the equation of the curve when $y = f(x)$ is reflected about a line $y = kx$, where k is a constant.

I shall attempt to solve this question here. I use algebraic method with the aid of the software TI Nspire CAS to solve it, the software helps me to shorten the time spent in doing all the tedious algebraic manipulation, without using software like this, I think I will not attempt to solve this question at all. Hence technology like this enables us to do more challenging question.

We shall use $y=e^x$ as an illustration here.

1. Sketch $f_1(x) = \tan\theta.x$, add in a slider for θ, where $\theta \in [0,360]$, θ is in degree.

2. Sketch $f_2(x) = e^x$, use the software TI Nspire to reflect $y = f_2(x)$ about the line $y = f_1(x)$. Note that we have to go through the following steps: reflect $y = f_2(x)$ about the x-axis, followed by rotation of 2θ anticlockwise about origin. See figure 1. Here we only get the locus of the reflection, but not the equation. We can explore around by changing θ using the software. See figure 2.

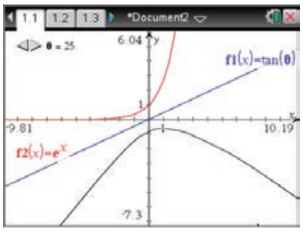


Figure 1

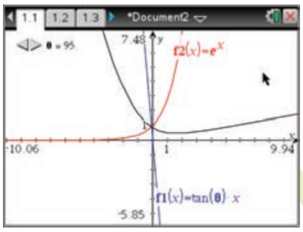


Figure 2

3. To find out the equation of the reflected curve, consider a general point $(t,f(t))$, when reflect about x-axis, we obtain $(t,f(t))$, under rotation of 2θ, we use matrix multiplication to get the coordinates of the point. See figure 3. As this software is interactive, changing the slider in figure 1 will automatically change the coordinates of the point. See figure 4. Note that the equations obtained are in parametric form:

$$x=\sin 2\theta . e^t + \cos 2\theta . t$$
$$y=\sin 2\theta . t - \cos 2\theta . e^t$$

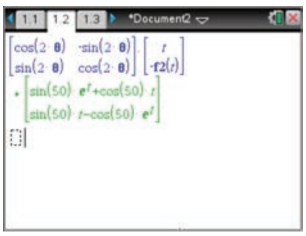


Figure 3

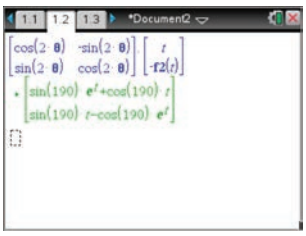


Figure 4

4. I use the software to do a parametric curve sketch. See figure 5.

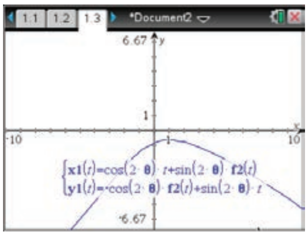


Figure 5

5. In order to find the Cartesian equation of the curve, I use the software to help me to solve. See figure 6. Which means the Cartesian equation is $e^{\cos 2\theta . x + \sin 2\theta . y} = \sin 2\theta . x - \cos 2\theta . y$

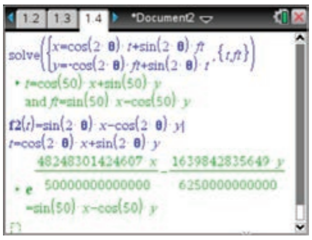


Figure 6

6. Again I use the software to key in the Cartesian equation obtained. See figure 7.

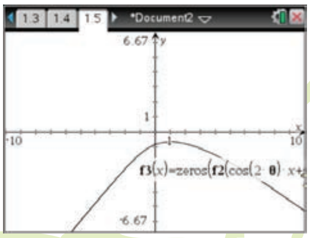


Figure 7

7. I go back to figure 1 and change the angle θ to be 45, I obtained the following: see figure 8.

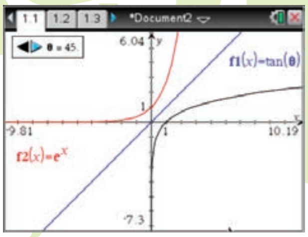


Figure 8

8. I go back to figure 1 and change the $f_1(x)$ to x^2 and θ to 25. See figure 9. The parametric equations of the reflection are:

$$x = \cos 2\theta . t + \sin 2\theta . t$$
$$y = -\cos 2\theta . t^2 + \sin 2\theta . t$$

And the Cartesian equation is:

$$[\cos 2\theta . x + \sin 2\theta . y]^2 = \sin 2\theta . x - \cos 2\theta . y$$

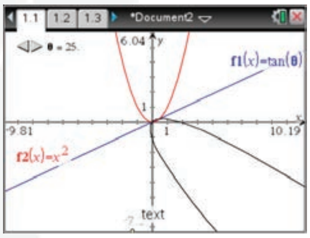


Figure 9

9. In general, when $y = f(x)$ is reflected about a line y= tan θ x, we obtained the parametric equations:

$$x = \cos 2\theta . t + \sin 2\theta . f(t)$$
$$y = -\cos 2\theta . f(t) + \sin 2\theta . t$$

And the Cartesian equation is:

$$f(\cos 2\theta . x + \sin 2\theta . y) = \sin 2\theta . x - \cos 2\theta . y \qquad (1)$$

I also attempt to prove the result in point 9 above by using coordinates geometry.

Given a function $y^1 = f(x^1)$, after reflected about the line $y = kx$, we want to find the equation $y = g(x)$.

Let (a,b) be a point on the curve $y^1 = f(x^1)$, let (x,y) be the point of reflection of (a, b) about the line $y = kx$. Using the fact that the midpoint of (x, y) and (a, b) lies on the line $y = kx$, and also the gradient of the line joining (x, y) and (a, b) is perpendicular to the line $y = kx$.

With the aid of TI Nspire CAS, we solve for simultaneously equations, see Figure 10:

$$-k(b-y) = a - x,\ b+y = k(a - x),$$

we get

$$a = \frac{2kx + (k^2 - 1)y}{k^2 + 1}, \quad b = \frac{2ky + (1 - k^2)y}{k^2 + 1}$$

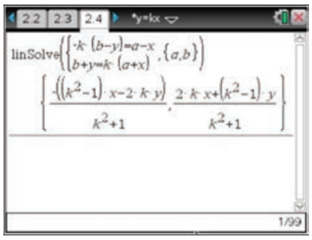


Figure 10

Sub a and b into $y^1 = f(x^1)$,

we have

$$\frac{2kx + (k^2 - 1)y}{k^2 + 1} = f\left(\frac{2ky + (1 - k^2)x}{k^2 + 1}\right) \qquad (2)$$

Using TI Nspire to sketch the graphs, see Figure 11, we shall use $y = x^3$ as an illustration here.

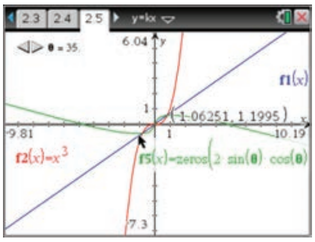


Figure 11

Since $\tan\theta = k$, which implies $\sin 2\theta = \frac{2k}{1+k^2}$, $\cos 2\theta = \frac{1+k^2}{1+k^2}$, it is obvious that equation (1) is the same as equation (2).

Ask Dr Maths Teaching

Secondary Mathematics

Question 1 :

Integration is the reverse process of differentiation. Since $\frac{d}{dx}(\ln x)=\frac{1}{x}$, so we would expect that $\int \frac{1}{x}dx = \ln x + c$ where c is an arbitrary constant. But some teachers insist that $\int \frac{1}{x}dx = \ln|x| + c$. Is there anything wrong with this argument?

Answer :

Note that for $\frac{d}{dx}(\ln x)=\frac{1}{x}$ to hold, $\mathbf{x}$ has to be positive in order for ln $\mathbf{x}$ to be defined.

In $\int \frac{1}{x}dx$, $\mathbf{x}$ can be either positive or negative.

Since $\frac{d}{dx}(\ln(-x))=\frac{1}{x}$ where $\mathbf{x} < 0$, so $\int \frac{1}{x}dx = \left\{ \begin{array}{ll} \ln x + c & \text{if } x > 0, \\ \ln(-x) + c & \text{if } x > 0. \end{array} \right\}$

$= \ln|x| + c$

Question 2 :

How do we prove that for a triangle ABC with AB = c, AC = b, BC = a, if $c^2 = a^2 + b^2$, then the triangle ABC is a right-angled triangle ?

Answer :

In fact, what you wish to prove is the converse of Pythagoras' Theorem.

We first construct a right angled triangle PQR such that PR = b, QR = a and $\angle$PRQ = 90°.

By Pythagoras' Theorem, PQ = $\sqrt{(PR)^2 + (QR)^2} = \sqrt{a^2 + b^2}$

By the given condition $c^2 = a^2 + b^2$, we conclude that PQ = c = AB.

Also, PR = AC = b, QR = BC = a.

Using the SSS congruency test, we can conclude that triangles ABC and PQR are congruent. Hence the corresponding angles are equal which implies that $\angle$ACB = $\angle$PRQ = 90°.

Question 3 :

When expressing an algebraic fraction, say $\frac{2x+3}{(x-1)(5-x)}$, in partial fractions of the form $\frac{A}{x-1}+\frac{B}{5-x}$, one method to find the values of A and B is by Cover-up Rule.

By Cover-up Rule, $A = \frac{2(1)+3}{5-1} = \frac{5}{4}$, $B = \frac{2(5)+3}{5-1} = \frac{13}{4}$.

Why does the Cover-up Rule work? When using the Cover-up Rule to find the value of A, we put $\mathbf{x}$ = 1 and when finding the value of B, we put $\mathbf{x}$ = 5. Isn't the algebraic fraction $\frac{2x+3}{(x-1)(5-x)}$ undefined when $\mathbf{x}$ = 1 or $\mathbf{x}$ = 5 ?

Answer :

From $\frac{2x+3}{(x-1)(5-x)} = \frac{A}{x-1}+\frac{B}{5-x}$, let $\mathbf{x} \neq 1$ and $\mathbf{x} \neq 5$.

Multiply both sides by $\mathbf{x}$ – 1.

Then $\frac{2x+3}{5-x} = A + \frac{B(x-1)}{5-x}$ ....(I)

Both $\frac{2x+3}{5-x}$ and $A + \frac{B(x-1)}{5-x}$ are continuous functions for all real values of $\mathbf{x}$ except $\mathbf{x}$ = 5.

Put $\mathbf{x}$ = 1 in (I) and we obtain $A = \frac{2(1)+3}{5-1} = \frac{5}{4}$.

Now observe that the value $\frac{2(1)+3}{5-1}$ is obtained from the algebraic fraction $\frac{2x+3}{(x-1)(5-x)}$ by covering up the term $\mathbf{x}$ – 1 in $\frac{2x+3}{(x-1)(5-x)}$ and putting $\mathbf{x}$ = 1 in the remaining expressions. Note that $\mathbf{x}$ – 1 is associated with the constant A in the partial fraction decomposition.

Similarly, the value of B can be obtained by covering the term 5 – $\mathbf{x}$ in $\frac{2x+3}{(x-1)(5-x)}$ and putting $\mathbf{x}$ = 5 in the remaining expressions. Note that 5 – $\mathbf{x}$ is associated with the constant B in the partial fraction decomposition.

A-Level Mathematics

Question 1 :

By formula, $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + C$. Since $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$, am I also correct to conclude that $\int \frac{1}{\sqrt{1-x^2}}dx = \cos^{-1}x + C$? But these 2 expressions, $\sin^{-1}\mathbf{x}$ and $-\cos^{-1}\mathbf{x}$, look so different! Please enlighten me.

Answer :

It can be proven that $\sin^{-1}\mathbf{x} + \cos^{-1}\mathbf{x} = \frac{\pi}{2}$ and substituting this relation in $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + c$, we obtain $\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x + c + \frac{\pi}{2} = -\cos^{-1}x + c_1$ where $c + \frac{\pi}{2} = c_1$.

So it is correct to say that $\int \frac{1}{\sqrt{1-x^2}}dx = -\sin^{-1}x + c$ and $\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x + c_1$.

The confusion arises because you use the same notation for the arbitrary constants.

We can evaluate the definite integral $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx$ using the two expressions and obtain the same numerical value.

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx = \left[\sin^{-1}x\right]_0^{\frac{1}{2}} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6},$$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx = \left[-\cos^{-1}x\right]_0^{\frac{1}{2}} = -\cos^{-1}\frac{1}{2} + \cos^{-1}0 = \frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{6}.$$

Question 2 :

When I ask my students to find the expansion of $\frac{1}{1-x-2x^2}$ in ascending powers of $\mathbf{x}$ up to and including the term in $\mathbf{x}^4$ and the range for which the expansion is valid, I expect them to write $\frac{1}{1-x-2x^2} = \frac{1}{(1-2x)(1+x)}$ as $(1-2x)^{-1}(1+x)^{-1}$ and find the respective expansions of $(1-2x)^{-1}$ and $(1+x)^{-1}$. One of my students did the following instead:

$$\frac{1}{1-x-2x^2}$$
$$= \left(1-\left(x+2x^2\right)\right)^{-1}$$
$$= 1 + (-1)(-1)\left(x+2x^2\right) + \frac{(-1)(-2)}{2!}(-1)^2\left(x+2x^2\right)^2 + \frac{(-1)(-2)(-3)}{3!}(-1)^3\left(x+2x^2\right)^3 +$$
$$\frac{(-1)(-2)(-3)(-4)}{4!}(-1)^4\left(x+2x^2\right)^4 + \ldots$$
$$= 1 + \left(x+2x^2\right) + \left(x+2x^2\right)^2 + \left(x+2x^2\right)^3 + \left(x+2x^2\right)^4 + \ldots$$
$$= 1 + x + 2x^2 + \left(x^2+4x^3+4x4\right) + \left(x^3+6x^4+\ldots\right) + \left(x^4+\ldots\right)..$$
$$= 1 + x + 3x^2 + 5x^3 + 11x^4 + \ldots$$

which coincides with the first few terms in the expansion obtained from $(1-2x)^{-1}(1+x)^{-1}$.

But his range of validity of $\mathbf{x}$ is $|x+2x^2|<1$ which is different from the range of validity of $\mathbf{x}$ for the expansion of $(1-2x)^{-1}(1+x)^{-1}$ which is $|x|<\frac{1}{2}$. Why?

Answer :

Your student is expressing $\frac{1}{1-x-2x^2}$ as a power series in $(x+2x^2)$. As such, his range of validity is $|x+2x^2|<1$ which upon simplification gives $-1<x<\frac{1}{2}$.

Your student's working is correct up to the step that says

$=1+(x+2x^2)+(x+2x^2)^2+(x+2x^2)^3+(x+2x^2)^4+\ldots$

and this is valid provided $-1<x<\frac{1}{2}$.

The next step which involves rearrangement of terms in an infinite series requires a restriction of the range of validity.

Note that

$1+(x+2x^2)+(x+2x^2)^2+(x+2x^2)^3+(x+2x^2)^4+\ldots=1+x+3x^2+5x^2+11x^4+\ldots$

does not hold for all values of $\mathbf{x}$ satisfying $-1<x<\frac{1}{2}$.

For example, let $x=-\frac{3}{4}$. Then one can check that

$1+(x+2x^2)+(x+2x^2)^2+(x+2x^2)^3+(x+2x^2)^3+\ldots$

$=1+\frac{3}{8}+\left(\frac{3}{8}\right)^2+\left(\frac{3}{8}\right)^3+\left(\frac{3}{8}\right)^4+\ldots$ converges

but

$1+x+3x^2+5x^2+11x^4+\ldots$

$=1+\left(-\frac{3}{4}\right)+3\left(-\frac{3}{4}\right)^2+5\left(-\frac{3}{4}\right)^3+11\left(-\frac{3}{4}\right)^4+\ldots$ diverges.

So it is not correct for your student to write

$=1+(x+2x^2)+(x+2x^2)^2+(x+2x^2)^3+(x+2x^2)^4+\ldots$

$=1+x+2x^2+(x^2+4x^3+4x^4)+(x^3+6x^4+\ldots)+(x^4+\ldots)+\ldots$

$=1+x+3x^2+5x^3+11x^4+\ldots$

even though the first 5 terms obtained by your student in his expansion coincide with the first 5 terms in the expansion of $(1-2x)^{-1}(1+x)^{-1}$.

However, if $\mathbf{x}$ is close to zero such that terms involving $\mathbf{x}^n$ and above for some positive integer n , say $\mathbf{x}^5$ in this case, can be neglected, then it is valid if your student writes as follows :

$$\frac{1}{1-x-2x^2}$$

$$=\left(1-(x+2x^2)\right)^{-1}$$

$$\approx 1+(-1)(-1)(x+2x^2)+\frac{(-1)(-2)}{2!}(-1)^2(x+2x^2)^2+\frac{(-1)(-2)(-3)}{3!}(-1)^3(x+2x^2)^3+\frac{(-1)(-2)(-3)(-4)}{4!}(-1)^4(x+2x^2)^4$$

$$=1+(x+2x^2)+(x+2x^2)^2+(x+2x^2)^3+(x+2x^2)^4$$

$$\approx 1+x+2x^2+(x^2+4x^3+4x^4)+(x^3+6x^4+\ldots+8x^6)+(x^4+\ldots+16x^8)$$

$$\approx 1+x+3x^2+5x^3+11x^4.$$

Question 3 :

Consider a box containing 5 red balls and 3 white balls. The balls are non-distinguishable except for their colours. Two balls are randomly drawn from the box without replacement.

Let B denote the event that of the two balls drawn, one is red and one is white.

To find B, we can either use

$P(B)=\frac{5}{8}\text{x}\frac{3}{7}\text{x}2=\frac{15}{28}$. or $P(B)=\frac{{}^5C_1\,{}^3C_1}{{}^8C_2}=\frac{15}{28}$.

Why do both methods work and why is there a need to multiply by 2 for the first method ?

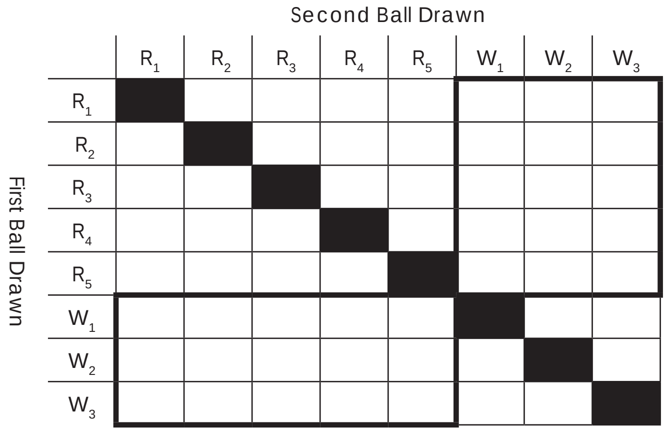
Answer :

For the first method, we are drawing the 2 balls sequentially, that is, drawing one ball after another. For the second method, we are drawing the 2 balls simultaneously.

In both cases, the balls are treated as being distinct.

Let R denote a red ball and W denote a white ball. Denote the set of 5 red balls by {R_1, R_2, R_3, R_4, R_5} and the set of 3 white balls by {W_1, W_2, W_3}.

First method of drawing the 2 balls sequentially :



The sample space has 56 elements.

To compute P(B), we can either draw a red ball first, followed by a white ball or draw a white ball first, followed by red ball.

The probability of drawing a red ball first, followed by a white ball is $\frac{5}{8}\text{x}\frac{3}{7}$, which is the same as the probability of drawing a white ball first, followed by a red ball which is given by $\frac{3}{8}\text{x}\frac{5}{7}$.

(You can check from the above that the event of drawing a red ball first followed by a white ball consists of 5×3 = 15 elements while the event of drawing a white ball first followed by a red ball consists of 3×5 = 15 elements as well.)

Hence $P(B)=\frac{5}{8}\text{x}\frac{3}{7}+\frac{5}{8}\text{x}\frac{3}{7}=\frac{5}{8}\text{x}\frac{3}{7}\text{x}2=\frac{15}{28}$.

Second method of drawing the two balls simultaneously :

The number of elements in this sample space is half the number of elements in the previous case, that is, $\frac{56}{2}=28$ elements, which can also be obtained using ${}^8C_2=28$.

The number of ways of drawing 1 red ball out of 5 and 1 white ball out of 3 is given by ${}^5C_1$ x ${}^3C_1$ $=15$

Hence $P(B)=\frac{{}^5C_1\,{}^3C_1}{{}^8C_2}=\frac{15}{28}$.

Also, $\frac{{}^5C_1\,{}^3C_1}{{}^8C_2}=\frac{5\text{x}3}{\frac{8\text{x}7}{1\text{x}2}}=\frac{5}{8}\text{x}\frac{3}{7}\text{x}2.$

The answers, obtained from these 2 different methods of looking at the same question, tally.

Contributions Invited:

You may email your contributions to the following:

- **mathsbuzz.ame@gmail.com** – for sharing of teaching ideas or research findings
- **askdrmathsteaching.sg@gmail.com** – for discussion and clarification of issues related to teaching and learning of mathematics