Maths Buzz

ASSOCIATION OF MATHEMATICS EDUCATORS | http://math.nie.edu.sg/ame/

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PRESIDENT'S MESSAGE

Dear AME members,

Thanks to your continuing support, AME celebrated her 20th anniversary in 2014. Membership has grown steadily throughout these 20 years to 105 life members, 421 ordinary members, and 167 corporate members. Your enthusiastic response is the motivation for the AME EXCO (we are the 11th EXCO!) to continue with efforts to organise activities that augment the important roles played by all mathematics educators in Singapore. We look forward to your continued support and participation in our activities.

2015 marks the fourth year that AME and SMS (Singapore Mathematical Society) will jointly organise the AME-SMS Conference. The theme for this year will be Developing 21st Century Competencies in the Mathematics Classroom. We hope that you will join us on 4 June 2015 at the NUS High School of Mathematics and Science, for a day of learning and update on recent developments in mathematics education.

This is also the fourth year that AME organises institutes for primary and secondary teachers who are keen on acquiring in-depth knowledge and skills in designing better learning experiences for their students. Dr Eric Chan will lead the primary institute (Engaging Pupils in Model-Eliciting Activities) while Dr Wong Khoon Yoong helms the secondary institute (Teaching Tricky Topics Through Thoughtful Techniques). Both AME institutes are scheduled to take place during the March school holidays.

Do you have suggestions for other AME activities? We look forward to your suggestions, as well as opportunities to partner with your schools or institutions.

Our two publications, The Mathematics Educator (TME) and Maths Buzz, continue to serve both local and international readers. TME publishes research in mathematics education and is now under the editorial leadership of Professor Berinderjeet Kaur. We welcome articles on your research findings. Maths Buzz reports on AME activities and shares mathematical ideas and teaching episodes. Math Buzz continues its production through the editorial work of two Dr Tohs, Dr Toh Tin Lam and Dr Toh Pee Choon. We welcome articles that share teaching ideas and also questions for Ask Dr Maths Teaching.

Low-Ee Huei Wuan
President,
AME (2014 – 2016)

AME EXCO
(2014-2016)

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Why is minus times a minus a plus?

Paul M. E. Shutler
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Introduction

Everyone finds negative numbers rather abstract and their operations difficult to explain, more so than similar operations involving fractions, such as why dividing by a fraction is the same as multiplying by its reciprocal. This difference is also apparent historically, since even the ancient Egyptians had a well worked out system of fractions more than 4,000 years ago, whereas the existence of negative numbers was not widely accepted by mainstream mathematicians until about 300 years ago. Our modern school syllabus also follows this historical order, with fractions taught as early as lower primary, but negative numbers generally delayed until secondary school.

From a purely mathematical point of view, that fractions should be easier to understand than negative numbers appears at first sight to be strange. Fractions are what you get if you insist on being able to divide any two whole numbers, not just multiply them, whereas negative numbers are what you get if you insist on subtracting any two whole numbers, not just adding them. Since multiplication and division are more complicated operations than addition and subtraction, surely fractions should be *more* difficult to understand than negative numbers, not the other way round.

Fractions do appear more complicated because they involve pairs of whole numbers, whereas negative numbers only involve a single whole number with a sign. This is because, even reducing a fraction m/n to its simplest form, numerator and denominator are generally both not 1, whereas the difference (m – n) of any two distinct whole numbers is always either a wholly positive or a wholly negative number. In this article we shall see how this apparent simplicity of negative numbers is in fact the cause of the problem, and that by choosing to regard negative numbers as differences between *pairs* of whole numbers, which we shall call *difference* pairs, rather than as a single whole number with a sign, their properties in fact become easier to understand.

Definitions

The simplest definition of a fraction, and the one normally used in primary schools, is as a number of equal parts of a whole, for example 6/8 means a whole unit divided into eight equal parts of which you take six. A more advanced definition is as an *uncompleted division*, for example 6/8 means 6 ÷ 8, or six units divided into eight equal parts, even if this does not immediately make sense as a whole number of units. The advantage of this second definition is that the notion of equivalent fractions is easier to motivate in terms of performing any part of the division which does make sense, for example 6/8 and 3/4 are equivalent since 6 ÷ 8 = 6 ÷ (2 x 4) = (6 ÷ 2) ÷ 4 = 3 ÷ 4.

The simplest definition of negative numbers, and again the one normally used in schools, is as points on the number line below 0, which allows you to make sense of subtracting a larger number from a smaller, for example 5 – 8 = –3, by counting downwards. By analogy with fractions, we could instead define a negative number as an *uncompleted subtraction*, that is, an expression of the form (m – n), even if this does not immediately make sense as a whole number. Again by analogy with fractions, we have a notion of equivalence, which is to perform any part of the subtraction which does make sense, for example 5 – 8 = 5 – (5 + 3) = (5 – 5) – 3 = 0 – 3 which we can identify as the negative number –3.

In more mathematical language, what we have done is to define the set of fractions to be the set of pairs of whole numbers, written suggestively as m/n, where any two pairs differing by a common factor top and bottom are counted as really being the same, for example 3/4 = 6/8 = 9/12. What we are proposing is to define the set of positive and negative numbers to be the set of pairs of whole numbers, written suggestively as (m – n), where any two pairs differing left and right by a constant amount are counted as really being the same, for example (5 – 8) = (4 – 7) = (3 – 6). Given any such *difference pair* (m – n), we can always cancel the smaller against part of the larger to give either (k – 0), which is the positive number +k, or (0 – k), which is the negative number –k, so this alternative definition in fact gives the same set as the usual definition.

Adding negative numbers

We multiply fractions by multiplying their numerators and denominators separately, that is, m/n x p/q = (m x p)/(n x q), for example 3/4 x 5/7 = (3 x 5)/(4 x 7) = 15/28. In terms of uncompleted division what this example is saying is that (3 ÷ 4) x (5 ÷ 7) = (3 x 5) ÷ (4 x 7) = 15 ÷ 28, which makes sense since we know that dividing by 4 and then by 7 is the same thing as dividing by 4 x 7 = 28. In fact, this formula for multiplying fractions is true and verifiable in terms of whole numbers in those cases where the divisions can actually be carried out, for example 12/4 x 8/2 = 96/8 because the left hand side is 3 x 4 = 12 while the right hand side is 96 ÷ 8 = 12 also.

Suppose by analogy we add difference pairs by adding left and right terms separately, that is, (m – n) + (p – q) = (m + p) – (n + q), for example (3 – 4) + (5 – 7) = (3 + 5) – (4 + 7) = (8 – 11). This makes sense in terms of uncompleted subtraction since we know that subtracting 4 then subtracting 7 is the same thing as subtracting 4 + 7 = 11. This formula for adding difference pairs turns out also to be true and verifiable in terms of whole numbers in those cases where the subtractions can actually be carried out, for example (10 – 7) + (6 – 3) = (10 + 6) – (7 + 3) since the left hand side is 3 + 3 = 6 while the right hand side is 16 – 10 = 6 also.

The usual addition of positive and negative numbers has to be split into four different cases: positive+positive, negative+negative, large positive+small negative, small positive+large negative. When we check that our addition of difference pairs gives the same result, we notice that it is not immediately obvious whether any given difference pair (m – n) represents a positive or a negative number, so our new addition of pairs should do all four cases at once. In fact this turns out exactly to be the case: e.g. (+7) + (–3) = (+4) becomes (7 – 0) + (0 – 3) = (7 + 0) – (0 + 3) = 7 – 3 = 4 – 0, also (+5) + (–9) = (–4) becomes (5 – 0) + (0 – 9) = (5 + 0) – (0 + 9) = 5 – 9 = 0 – 4, and it is easy to check that the other two cases give the same results as well.

Subtracting negative numbers

We divide by a fraction by multiplying by its reciprocal, that is y ÷ (m/n) = y x (n/m), where y represents any other fraction. For example y ÷ (3/4) = y x (4/3), which in terms of uncompleted division says y ÷ (3 ÷ 4) = y x (4 ÷ 3), so the ÷3 remains but the ÷ (÷4) becomes x4, which makes sense because if we know that the number we are dividing by is itself reduced by a factor 4, the answer becomes 4 times larger. This formula for dividing by fractions is true and verifiable in those cases where the divisions can actually be carried out, for example 12 ÷ (10 ÷ 5) = (12 x 5) ÷ 10, since the left hand side is 12 ÷ 2 = 6 while the right hand side is 60 ÷ 10 = 6 also.

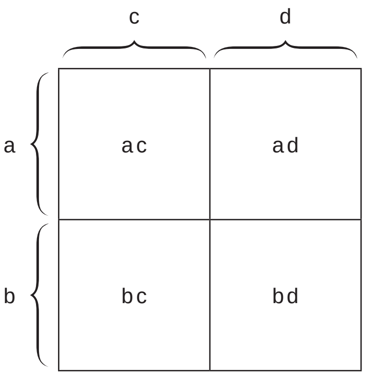
Suppose, by analogy with fractions, we define subtraction by a difference pair to be addition of the "reciprocal" pair, in other words the pair with the order swapped, y – (m – n) = y + (n – m). This swapping of the order is usually called *negation*. For example y – (3 – 7) = y + (7 – 3), which in terms of uncompleted subtraction says that the –3 remains while the –(–7) becomes +7, which makes sense because if we know that the number we are subtracting is itself reduced by 7 units, the answer becomes 7 units larger. Again, this formula turns out to be true in those cases where the subtractions can actually be carried out, for example 10 – (5 – 2) = (10 + 2) – 5, since the left hand side is 10 – 3 = 7 while the right hand side is 12 – 5 = 7 also.

To check that subtracting difference pairs this way gives the usual subtraction of positive and negative numbers we note that the latter is again split into different cases: subtracting a positive means adding a negative, and subtracting a negative means adding a positive. As before, since we cannot immediately tell whether (m – n) is positive or negative, our subtraction formula in fact gives both cases at once: for example y – (+5) = y + (–5) becomes y – (5 – 0) = y + (0 – 5), while y – (–7) = y + (+7) becomes y – (0 – 7) = y + (7 – 0). From a difference pairs point of view, double negation means swapping then swapping back, which clearly leaves the pair unchanged, just as does double reciprocation, and this is a much easier way to understand why –(–y) = y than using the normal definition of positive and negative numbers.

Multiplying negative numbers

If we define negative numbers as difference pairs (m – n), then to know how to multiply them we need to know how subtraction and multiplication interact, analogous to the way in which addition and multiplication interact in the *distributive law* (a + b) × (c + d) = ac + bd + bc + ad. Provided we adopt a symmetrical definition of multiplication, that m x n is the number of elements in a matrix with m rows and n columns, the proof of the distributive law is not hard and is illustrated in Figure 1. We can motivate this using examples such as (10 + 2) x (10 + 3) = 100 + 6 + 20 + 30, which is true since the left hand side is 12 × 13 = 156, and in fact the four terms corresponds to the steps involved in long multiplication.

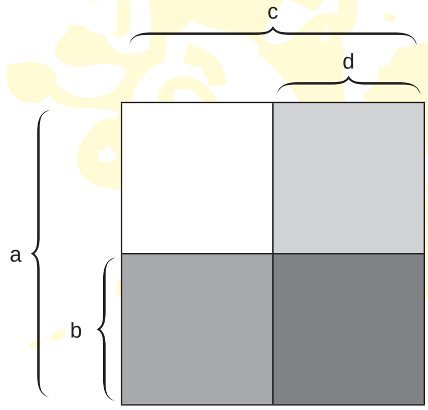
Figure 1. Usual form of the distributive law: (a + b) × (c + d) = ac + bd + bc + ad.



To multiply difference pairs we need the version of the distributive law where both terms on the left hand side are subtractions, that is (a – b) × (c – d) = (ac + bd) – (bc + ad). Again the proof is not hard and is shown in Figure 2, the essential point being that when the two shaded rectangles are subtracted from the square, the bottom right hand corner is double subtracted, so the bd term must be added to ac to cancel this out. This is also easily motivated by simple examples such as 9 x 9 = (10 – 1) x (10 – 1) = (100 + 1) – (10 + 10) = 81, which also makes it more convincing that the bd term must be added, not subtracted.

If we assume that this alternative form of the distributive law still holds true even in those cases where the subtractions are uncompletable, then it becomes a statement of how to multiply positive and negative numbers. Again, what are normally counted as separate cases are all done at one go as follows: that a positive times a negative is a negative, e.g. (+3) x (–2) = –6, becomes (3 – 0) x (0 – 2) = (3 x 0 + 0 x 2) – (0 x 0 + 3 x 2) = (0 – 6), while that a negative times a negative is a positive, e.g. (–2) x (–3) = (+6), becomes (0 – 2) x (0 – 3) = (0×0 + 2×3) – (2 x 0 + 0 x 3) = (6 – 0), which answers the question which we posed at the start of this article.

Figure 2. Alternative form of the distributive law (a – b) × (c – d) = (ac + bd) – (bc + ad) where the darker shaded rectangle represents the term bc while the lighter shaded rectangle represents ad.



Conclusion

The usual definition of negative numbers is difficult to work with because it is *abstract*, that is, it creates a completely new set of "negative" numbers and we then have to find out how they interact with each other and the "positive" numbers with which we began. The alternative definition we are proposing is easier because it is *synthetic*, that is, the new numbers are just arithmetic *expressions* involving ordinary whole numbers, and their properties follow from the properties which the operations involved already have. For example, subtracting a negative means adding a positive because subtracting less means adding more, which is true even for whole numbers.

Although abstract definitions may seem a good idea, since most mathematics at the higher level is taught abstractly anyway, the historical origins of most modern mathematics are synthetic. Negative numbers, for example, appeared first in China, when methods which worked well for solving two equations in two unknowns were applied to three equations in three unknowns and found to produce "negative" expressions of the form (m – n), even though all the equations and all the solutions involved only positive whole numbers. Provided these expressions were manipulated according to the normal rules of addition and subtraction, in the end it was found that all the negative numbers disappeared, leaving behind only positive solutions.

One practical reason why fractions can be taught successfully at the primary level is the existence of good physical models, that is, if whole numbers in fact represent quantity of physical substance, such as bread or pizza, then they can be subdivided into smaller pieces. The number line model for positive and negative numbers is not so good because it defines numbers as positions on the line, but to explain addition and subtraction we also need to think of numbers as movements along the line, and children can be confused by the difference between the two. The Chinese physical model in this case was credit and debt, that is, you can physically withdraw more money than is currently in your bank account because it is sitting on top of a large quantity of the bank's own money, and perhaps research could be done to see if this model works better.

Solving Word Problems using Quadratic Equations

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Under the topic of "Solving word problems that are reducible to quadratic equations", a common observation among teachers is that some students struggle with translation of the statements in the "word problems" to equivalent equations. The frequently-used trajectory can be summarized as such: Teacher demonstrates the steps involved in translating statements to equations over different types of word problems; students can usually follow the steps; but when asked to do it on their own, they are 'stuck', especially when confronted with an unfamiliar type of "word problem". The usual response by teachers to such student difficulty is more demonstration and more fine-grained breakdown of steps with the intent of making the skill acquisition process for students more gradual. In this article, we propose a different approach.

We think the problem students encounter is not merely that of lacking familiarity with the different types of word problems; more fundamentally, it is the lack of opportunity for authentic exploration of the word problems – a necessary step for students to make sense of the problems and to appreciate the power of the algebraic approach. In other words, we need to "prepare the ground" so that when the algebraic method is "planted", it will "take root"—students will receive it and learn it better instead of seeing it as a method forced upon them. In particular, we attempted the MProSE problem solving approach. [MProSE is a research project funded by the Office of Educational Research, NIE. For more details about the project, the reader may refer to Toh, Quek, Leong, Dindyal, Tay (2011) and Leong et al. (2014)]

A key feature of this method is the use of the "Practical Worksheet", which consists of four sections that correspond to Polya's (1945) stages: Understand the Problem (UP), Devise a Plan (DP), Carry out the Plan (CP) and, Check & Expand (C/E). A compressed version of the Practical Worksheet can be found in the Appendix. Under UP, students explore the givens of the problem with a view of understanding the requirements and conditions; this exploration prepares them to consciously devise a plan (DP) to attack the problem; under CP, the students carry out the plan – and may loop back to UP and DP if unsuccessful in attempting the first plan; finally, under C/E, students check the correctness of the solution and seek extension to the original problem in order to have a deeper understanding of the structure of the problem and the suitability of the solution.

We anticipate that C/E is particularly unnatural for our students – when a solution is obtained, students have a tendency to see that as the end of any effort on the problem; but this habit of stopping with the first solution misses out on opportunities to learn problem solving extensions and alternative solutions which are ways to mine gems from the problem solving endeavour. We think that C/E lends itself well in making links in the topic: The link between students' informal approaches and the algebraic approach; the link between students' initial ad hoc method and a more generalized method for solving such problems.

For this topic on quadratic equations, instead of being taught a method of solving word problems right from the start, students were given time to attempt such a word problem on the Practical Worksheet. In so doing, they were given the opportunity to explore the word problem and hence figure out its underlying structure. The lessons in this topic were thus reorganized in this way:

Lesson 1: Solve a word problem reducible to quadratic equation with a Practical Worksheet.
Students are expected to use other methods such as systematic listing to solve the problem. Under C/E, the link to the algebraic method and thus the motivation to learn the solution to quadratic equations can be made.

Lesson 2: Solve a quadratic equation.
Students are to be taught the 'zero product rule' as the main argument behind the steps in solving quadratic equations by factorization. They then practise the method to gain fluency.

Lesson 3: Solve another given word problem using a Practical Worksheet.
Students are expected to use the resources gathered, both the experience in Lesson 1 on using the algebraic method as well as the method of solving quadratic equations in Lesson 2, to make productive attempts at solving the problem in this lesson. Under C/E, students can consider generalising a standard procedure for solving "word problems" that are reducible to quadratic equations.

Lesson 4: Apply the general procedure abstracted in Lesson 3 to solve other "word problems".
This sequence of lessons (module) was carried out with minor variations among different teachers, in all the Secondary Two Express classes in Bedok South Secondary School.

Observation of Enactment

We were able to observe the two lessons – Lessons 1 and 3 – where the Practical Worksheet was used in one class. Here, we discuss two episodes that illustrate the link we intended to make in the module.

In Lesson 1, the following problem was presented to the students:

"A company wants to employ as many workers as it can afford to complete a project within a short timeframe. If the company pays each worker $6 per hour, there are only 30 applicants for the job. However, the company needs more workers. It is known that for every $1 increment in the hourly pay, it will attract 2 more applicants for the job. The company can only afford a maximum of $504 per hour in total. How much should the company offer to pay per hour in order to attract the maximum number of workers?"

Since they had no prior learning about techniques of solving quadratic equations, a large majority of the students attempted the problem using some form of systematic listing, as we expected. The teacher invited a student to write her solution within the organisational frame as shown in Figure 1.

Figure 1. Systematic listing for the problem in Lesson 1.

	Pay ($)	Number of Applicants	Total Amount ($)
	6	30	180
	7	32	224
	-	-	-
	-	-	-
	-	-	-
Line (1)	12	42	504
	-	-	-
	-	-	-
Move (2):	-	-	-
(n – 6)	n	**2 (n – 6)** 2 (n – 6) + 30 = 2n – 12 +30 = 2n + 18	6572

Through repeated trials of entries into the table, the student was able to "hit" the right combination that yielded the total cost of $504 (as shown by *Line (1)* in Figure 1). At this point, the teacher used the C/E part of the Practical Worksheet to link to quadratic equation by asking a follow-up extension question, "What if the total is changed to $6572?" After a brief discussion through which the students could see that it was tedious to continue listing, the teacher entered another row into the table with these entries: "n", "2n + 18", and "6572", in this order to lead to the quadratic equation, "n (2n + 18) = 6572" (See *Move (2)* in Figure 1). When $n is paid per hour, the difference with $6 is $(n – 6), the difference in number of workers (beyond 30) that the rate would attract is therefore 2(n – 6). Hence, the number of workers the company can attract when the hourly rate is $n is 2(n – 6) + 30 = 2n + 18. This results in the equation "n (2n + 18) = 6572". This provided the motivation to learn the technique of solving quadratic equation – the focus of Lesson 2.

In Lesson 3, the following problem was given:
"Four consecutive even numbers are such that the product of the smallest and the largest is 186 more than the sum of the other two. What are the four numbers?"

Compared to the problem in Lesson 1, more students in Lesson 3 attempted the problem using the algebraic method. The teacher again used the C/E part of the worksheet to make another important link. In this case, the link was to convert the informal algebraic approach taken by the students to a more formal method that they can use in subsequent word problems. The intended link is illustrated by the teacher's presentation on the whiteboard, as shown in Figure 2.

Figure 2. Converting the informal algebraic approach to a method.

Working	Methodising
Let the first number be x.	Step 1: Determine the variable and let it be x.
Therefore, the four numbers are x, x+2, x+4 and x+6.	Step 2: Express the other variables in terms of x.
Product of the smallest and largest number = x (x + 6) Sum of the other two numbers = (x + 2) + (x + 4)	Step 3: Establish the relationships between x and the other terms based on the word phrases.
x(x + 6) = 186 + [(x + 2) + (x + 4)]	Step 4: Form an equation from the key sentence.
$x^2 + 4x - 192 = 0$	Step 5: Simplify the equation and equate it to zero.
x = -16 or x = 12	Step 6: Solve for x.

Discussion

It is a pity that the prevailing habit of problem solving is this: "As soon as I solve a problem, I move on to solve another problem". This is the kind of disposition that Polya's C/E was meant to counter. Instead of quickly moving away from the problem once it is deemed "solved", C/E inculcates the habit of stopping and mulling over the learning gems that can be picked up from a deeper contemplation about the structure of the problem and its solution. It involves processes such as checking, making extensions, and modifying solution strategy.

Although many Mathematics teachers we meet agree with C/E, a common add-on response is that "it is difficult to incorporate …". We are encouraged to see that minor modification to how a module is structured – such as the one we illustrated here in Bedok South Secondary School on the topic of word problems reducible to quadratic equations – can open up opportunities for C/E to be exemplified. In the process, we also introduce another conception on how C/E can look like in regular teaching of Mathematics: C/E as making links to what we intend to teach.

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The Practical Worksheet
(Modified Design by Bedok South Secondary School)

Mathematics Practical Worksheet

Session: **POD/HW**

Instructions
1. You may proceed to complete the worksheet doing stages I – IV.
2. If you wish, you have 15 minutes to solve the problem without explicitly using Polya's model. Do your work in the space for Stage III.
   - If you are stuck after 15 minutes, use Polya's model and complete all the stages I – IV.
   - If you can solve the problem, you must proceed to do stage IV – Check and Expand
   - You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.

Problem Of the Day :

Stage I: Understand the Problem

(a) Write down your feelings about the problem.

(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.

(c) How are you trying to understand the problem? State the heuristics you used.

Stage II: Devise a Plan

(a) Write down the key concepts that might be involved in solving the problem.
(b) Do you think you have the required ressources to implement the plan?

(c) Write out each plan concisely and clearly.

Stage III: Carry Out The Plan

(I) Write down in the Control column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.

(II) Write out each implementation in detail under the Detailed Mathematical Steps column.

Detailed Mathematical Steps	Control

Stage IV: Check and Expand

(a) Write down how you checked your solution.

(b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of.

(c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Dialogue between Student Teacher DTF and Teacher JAF

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This fictitious (but not altogether strange) story is based on actual observations of students' in-class work and the authors' perceptions of still-common ways of teaching mathematics formula. The authors would like to thank the Mathematics department of Bukit View Secondary.

How do you teach your students mathematics formulas?

DTF

Show them the formula then just apply formula to the textbook questions.

JAF

Do you not at least attempt to show your students how the formula is derived?

DTF

No use. Exams don't require that. Anyway, students are also not interested. They just want to know how to apply rules to do the questions.

JAF

Okay. Maybe we discuss over one example, say, the "laws" of indices. Take $a^m \times a^n = a^{m+n}$ for positive integers m and n.

Do you think this kind of development will help?

$$2^7 \times 2^6 = \underbrace{2 \times 2 \times \ldots \times 2}_{7} \underbrace{\times 2 \times 2 \times \ldots \times 2}_{6}$$

$$= \underbrace{2 \times 2 \times \ldots \qquad \times 2}_{7+6}$$

$$= 2^{7+6}$$

DTF

What for? They see it one time then they never use it again. Subsequently, they just apply the formula $a^m \times a^n = a^{m+n}$ without even needing to know this 'long story'.

JAF

This is precisely why I wanted to discuss with you. I thought so too until I observed a few 3NA lessons on this topic in Bukit View Secondary ...

DTF

What happened there?

JAF

The teacher spent one lesson just doing "coding" and "decoding". She said coding was like a contracted form of something longer. Like "LOL" is a coded form for "laugh out loud", 2^m is a coded form of $\underbrace{2 \times 2 \times \ldots \times 2}_{m}$.

DTF

What a waste of time!

JAF

I was wondering too. But I stayed on to observe. She then presented this as a problem to solve: "Write in as few symbols as possible $2^7 \times 2^6$." Further down the page, the students were prompted to extend what they find to $2^{18} \times 2^{14}$; then a final question was stated as, "Have you discovered a general rule?"

DTF

What was surprising was that many students, through decoding and coding, were able to get to the general rule of "adding indices" on their own with the teacher's encouragement. That sense of satisfaction on some of their faces was heartwarming to me.

DTF

Hmm ... I can almost imagine it. But, sorry to throw a spanner in the works: this is still one-off. I don't see how her coding/decoding can help the students in subsequent development of the topic.

JAF

Yes. I can understand this concern. I share it too. But I had a second surprise. The teacher introduced a similar problem but for $2^7 + 2^6$. Can you guess how it worked out?

DTF

Students would have been stuck.

JAF

Indeed, most can't make good progress. But the teacher use it to caution against careless rule-application without checking required conditions, such as $2^7 + 2^6 = 2^{13}$, and to highlight the importance of going back to coding/decoding when stuck:

$$2^7 + 2^6 = \underbrace{2 \times 2 \times \ldots \times 2}_{7} + \underbrace{2 \times 2 \times \ldots \times 2}_{6} \quad \text{[decode]}$$

$$= 2 \text{ x } \underbrace{(2 \times 2 \times \ldots \times 2)}_{6} + \underbrace{(2 \times 2 \times \ldots \times 2)}_{6} \quad [\text{like } 2a + a]$$

DTF

$= 3 \times \underbrace{(2 \times 2 \times \ldots \times 2)}_{6}$ [2a +a = 3a]

$= 3 \times 2^6$ [code]

DTF

Hmm … Must say this is quite nice …
But this kind of question is not standard textbook question.

JAF

I am coming to that. In the next lesson, I observed students tackling textbook questions that were initially not familiar to them. The first of its kind was $3^5 \times 9^3$.

DTF

Quite standard. I will teach them to do:

$3^5 \times 9^3 = 3^5 \times (3^2)^3$
$= 3^5 \times 3^6$ [apply $(a^m)^n = a^{mn}$]
$= 3^{11}$ [apply $a^m \times a^n = a^{m+n}$]

JAF

What I saw one student did was:

$3^5 \times 9^3 = 3^5 \times (3^2)^3$
$= 3^5 \times (3 \times 3) \times (3 \times 3) \times (3 \times 3)$
$= 3^{11}$ [after visibly counting the number of "3"s by mouthing "five", "six" …]

DTF

Apparently, quite a few students applied some form of decoding. The teacher noticed this and presented the solution above on the board. The solution you showed was also presented for the purpose of comparison. Looks like the message the teacher was sending to students is this: Use the "laws" if it works well for you; but if you are stuck, fallback to decoding/coding to make sense. To be honest, I think it is quite nice and a fairly balanced approach.

DTF

I must say I really didn't expect the students to do it in the way you showed.

JAF

Looks like there is some real value in developing the formula using coding/decoding. But how did the teacher deal with students of different pace of development ... some use formula while others still decode/code.

JAF

The teacher did not seem to be bothered about that. The thinking is: let the students decode/code for as long as he/she likes; do not 'force' them into formula-application mode; when they are ready, they will cross the boundary because the human mind is such that we want to economize.

DTF

Also, I can imagine that, even for students who are already formula-fluent, there will be situations where they need to fallback to decoding/coding. Like when they are stuck with unfamiliar problems.

JAF

I can see you are picking up their language of "fallback" ... I used to also think "just apply formula" but after what I observed in Bukit View Secondary, I think it works better, at least for the 3NA class I observed, when students are given the tools to develop the formula and given the time to "play with it" before the formula can be used fluently as a "chunk".

DTF

Sounds good. [Sigh] But in reality, we don't have time lah

JAF

[Student Teacher DTF doesn't know what to say to Teacher JAF at the moment. But he is determined to try some of the things he thinks will really help his students. Perhaps he will return with a sequel to this dialogue ...]

Marian Kemp (*A Tribute*)

The Association of Mathematics Educators (AME) pays tribute to a dear colleague who left us on 1st October 2014.

Marian contributed significantly towards the development of mathematics teachers in Singapore through the workshops and lectures she gave at the annual conferences of the Association. She also actively contributed towards the yearbooks of AME.

As a teacher, Marian taught secondary school mathematics in England, Jamaica, the Bahamas, Nigeria and Australia for almost twenty years across all ages and courses, providing her with an exceptionally rich background as a teacher educator. In a 30-year career at Murdoch University in Perth, Western Australia, Marian fulfilled several roles. She worked at first with pre-service primary and secondary mathematics teachers on campus and in local schools. Then as Murdoch's Numeracy Tutor, she helped students address the mathematical needs of their studies, and worked with staff to ensure that the significance of mathematical thinking in many course was understood and addressed. She also helped students studying undergraduate mathematics with their coursework, which required her to work well in many different areas of mathematics and statistics. In her last few years, she assumed the role of Head of the Student Learning Centre and then the senior administrative post of Director of Student Life and Learning. Although these positions were not concerned with mathematics education, she continued to teach and conduct research in mathematics education while undertaking them.

Marian's doctoral work was concerned with the idea of numeracy, focusing on the role of mathematics in everyday affairs. Those who attended her workshops at the AME conference will recall various examples of this, such as studying food packets to see how information was communicated and could be used, or examining tables and graphs to understand modern Singapore life. Her Five-Step Framework was especially important for such work, and found to be very helpful by teachers. She also developed considerable expertise with and interest in the role of technology in schools, especially calculators, and collaborated to produce some books, several publications in journals and conduct workshops at conferences to help teachers appreciate how important hand-held technologies are for school mathematics.

Marian was recognized as a wonderful teacher in many ways, including with a teaching excellence award at Murdoch University and with a prestigious national award for her effective promotion of numeracy. She was an active contributor to several professional societies in Australia, and a regular contributor to conferences for mathematics teachers in Australia and nearby countries. But her lasting legacy will be the many ways in which she helped struggling students develop both expertise and confidence in their own mathematics, rather than learning to imitate their teachers. She led both primary and secondary teachers by example in many countries to work quietly, effectively and warmly with their students and so was much loved by both students and colleagues, and will be sadly missed.

Working with local teachers at the AME-SMS Conference 2012

Contributions Invited:

You may email your contributions to the following:

- **mathsbuzz.ame@gmail.com** – for sharing of teaching ideas or research findings
- **askdrmathsteaching.sg@gmail.com** – for discussion and clarification of issues related to teaching and learning of mathematics