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Fallacies about the Derivative of the Trigonometric Sine Function

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In this paper, several fallacies about the extension of the formula $\frac{d}{dx}(\sin x) = \cos x$ to the erroneous formula $\frac{d}{dx}(\sin x°) = \cos x°$ are discussed. In a Commognitive Theory Framework, misconceptions by 'newcomers' can be traced to the use of the word "unit".

Keywords: derivative, trigonometric sine function, fallacies

Introduction

Calculus is a difficult topic of mathematics for both pre-university and undergraduate students (e.g. Amit & Vinner, 1990). Misconceptions are also common among practicing school teachers (e.g. Toh, 2009). At the school level, the version of calculus taught, which was termed "informal calculus" (Tall, 1993), usually dispenses with a rigorous treatment of the limit concept. This is not surprising as it has been informed by research that school students have difficulty in understanding the limit concept (e.g. Juter, 2006; Tall, 1993). School calculus, avoiding a rigorous treatment of limits, thus inevitably begins with a list of differentiation formulae for students to commit to memory. Developing students' procedural knowledge without the corresponding conceptual knowledge could easily lead to erroneous over-generalization of differentiation formulae. In the light of the limitations of school calculus, mathematics educators should at least ensure that mathematics teachers do not make fundamental mistakes in the teaching of calculus. One such mistake that we wish to discuss and ameliorate in this article is the extension of the formula $\frac{d}{dx}(\sin x) = \cos x$ to the erroneous formula $\frac{d}{dx}(\sin x°) = \cos x°$.

Differentiation of trigonometric functions

Finding the derivatives and antiderivatives of trigonometric functions is difficult for even undergraduate students (e.g. Siyepu, 2015). These difficulties are due to their lack of both procedural knowledge, as when they fail to carry out manipulations or algorithms, and their lack of conceptual knowledge, as when they fail to grasp the concepts in a problem. Difficulty with calculus due to students' inadequate algebraic manipulation skills and a relational understanding of calculus concepts is unfortunately common (e.g. Ng & Toh, 2008). In this note, several parts of a fallacy about the derivative of the trigonometric sine function will be presented and refuted.

In the next section, we will detail three fallacies that have arisen from misconceptions which we have encountered among a number of students and teachers. We will explain within a commognitive framework (Sfard, 2008) how these misconceptions occur and then propose a way to explain the correct concept by dealing with the commognitive conflict, that is, a "situation that arises when communication occurs across incommensurable discourses" (Sfard, 2008, p. 296).

Fallacies about the derivative of the trigonometric sine function

To avoid misleading the readers, all the fallacies presented in this paper will be italicized.

The independent variable matters

<u>*Fallacy 1*</u>

$\frac{d}{dx}(\sin x) = \cos x$ *is true regardless of whether x is measured in radians or degrees. For example, at the point when x =*$\frac{\pi}{4}$*, the gradient will have the value of* $\cos\frac{\pi}{4}$*. However, since* $\cos\frac{\pi}{4} = \cos 45°$*, the statement* $\frac{d}{dx}(\sin x°) = \cos x°$ *must also be true. The key idea is that "x" in "*$\frac{d}{dx}$*" and "x" in* $\sin x$ *are in the same unit (radian, degree or gradian).*

That the formula $\frac{d}{dx}(\sin x) = \cos x$ is true means that the independent variable x can be replaced by another symbol, e.g. $\frac{d}{du}(\sin u) = \cos u$, in which the independent variable is denoted by u. For example, if $u = 2x$, we will have $\frac{d}{d(2x)}(\sin 2x) = \cos 2x$. To reiterate from another point of view, we may see that the 'original' variable x is replaced by $2x$. However if dx is reinstated as the 'denominator', the statement $\frac{d}{dx}(\sin 2x) = \cos 2x$ is no longer true. In this case, the expression $\frac{d}{dx}(\sin 2x)$ needs to be simplified using the Chain Rule of calculus (giving the answer $2\cos 2x$). Along the same line of argument, the statement $\frac{d}{dx°}(\sin x°) = \cos x°$ is true when the variable x is replaced by $x°$ but if dx is reinstated as the 'denominator', $\frac{d}{dx}(\sin x°) = \cos x°$ is demonstrably untrue.

Note that $x°$ represents a scaling of the angle in radian, i.e. $x° = \frac{\pi}{180}x$. Hence, we can obtain $\frac{d}{dx}(\sin x°)$ by using the Chain Rule of calculus, noting that $\frac{d}{dx}x° = \frac{\pi}{180}$. Hence $\frac{d}{dx}(\sin x°) = \frac{d}{dx°}(\sin x°) \cdot \frac{dx°}{dx} = \frac{\pi}{180}\cos x°$. This is not different from our usual method of the common advice to convert angle to radian measures as

$$\frac{d}{dx}(\sin x°) = \frac{d}{dx}(\sin\frac{\pi x}{180}) = \frac{\pi}{180}\cos\frac{\pi x}{180} = \frac{\pi}{180}\cos(x°).$$

One can see the commognitive conflict that results from the use of the word "unit" by a newcomer and the word 'variable' by the old-timer. Conflict resolution here will succeed if we make sense of other people's talk and customize our discourse to theirs (Sfard, 2008). In this case, the newcomer 'sees' the unit completely within the 'x' and if so, the old-timer customizing her discourse to include this view, would acknowledge that "$\frac{\mathrm{d}}{\mathrm{dx}}(\sin x) = \cos x$, no matter what the unit of x" is true. Following from this common discourse, instead of using the word variable, the old-timer would continue by saying that to ensure that all units are the same, $\frac{\mathrm{d}}{\mathrm{dx}}(\sin x) = \cos x$ must transfer completely to $\frac{d}{dx^\circ}(\sin x^\circ) = \cos x^\circ$ and not $\frac{d}{dx}(\sin x^\circ) = \cos x^\circ$.

Verification with a graph plotter

<u>*Fallacy 2*</u>
The value on the right hand side of $\frac{d}{dx}(\sin x^\circ) = \frac{\pi}{180}\cos x^\circ$ can never be larger than $\frac{\pi}{180}$, no matter the value of x. But the derivative of sine should vary between the value of -1 and 1 inclusive. Thus, the statement $\frac{d}{dx}(\sin x^\circ) = \frac{\pi}{180}\cos x^\circ$ must be incorrect.

A graph plotter can be used to disprove Fallacy 2. Figures 1 and 2 show the graphs of $y = \sin x$ and $y = \sin x^\circ$ and the tangent lines at the points where $x = \frac{\pi}{6}$ and $x = 30$, respectively.

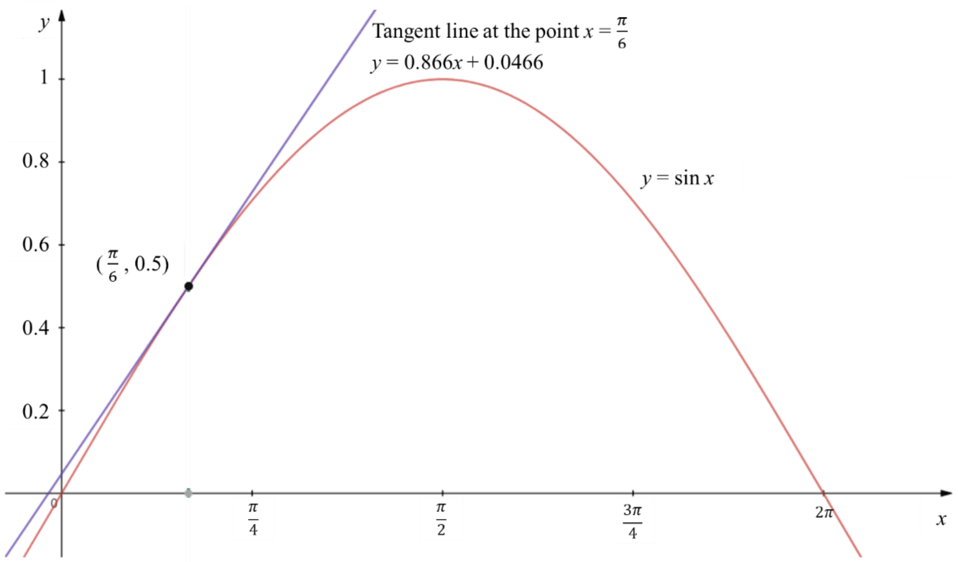

Figure 1. Diagram of $y = \sin x$ and the tangent at the point $x = \frac{\pi}{6}$ has a slope of 0.866 (3 s.f.)

In Figure 1, the gradient of the graph $y = \sin x$ (in radians) plotted against the variable x at the point $x = \frac{\pi}{6}$ has the value of $\frac{\sin\frac{\pi}{6} - 0.0466}{\frac{\pi}{6}} = 0.866$ (to 3 s.f.). On the other hand, in Figure 2, the gradient of the graph $y = \sin x^o$ plotted against the variable x at the point $x = 30$ has an approximate value of $\frac{\sin 30^\circ - 0.0466}{30} = 0.0151$ (to 3 s.f.), a much smaller value than the previous one. Here, the values of the slopes are obviously different.

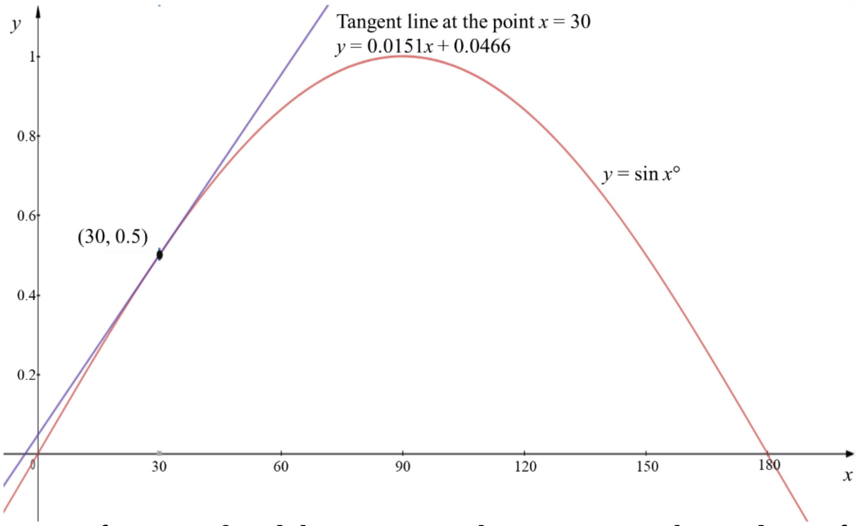

Figure 2. Diagram of $y = \sin x^{\text{o}}$ and the tangent at the point $x = 30$ has a slope of 0.0151 (3 s.f.)

The graphs of $y = \sin x$ and $y = \sin x^{\text{o}}$ are plotted on the same axes in Figure 3. It can be seen that the latter graph is much more gentle compared with the former. Indeed, we can now probably understand why the gradient of the curve $y = \sin x^{\text{o}}$ does not exceed $\frac{\pi}{180}$. The figure also accentuates the visual perception that the gradient (≈ 0.8660) at the point when $x = \frac{\pi}{6}$ on the graph of $y = \sin x$ is steeper compared to the gradient (≈ 0.0151) at the point when $x = 30$ on the graph of $y = \sin x^{\text{o}}$.

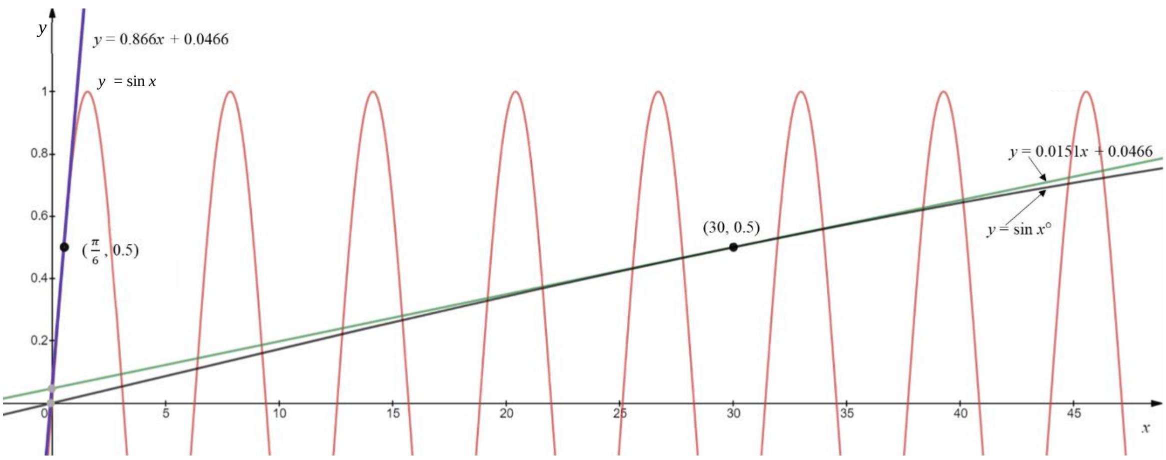

Figure 3. Diagram of $y = \sin x^{\text{o}}$ (a quarter cycle) and $y = \sin x$ (with many cycles) from x = 0 to $x = 45$.

It is good practice pedagogically to apply Dienes' Variability Principle (Dienes, 1971) to allow for accommodation of a new idea. Thus, the old idea that a sine curve will always have gradients between -1 and 1 will best be challenged by more than one example. Using the previous example of $\frac{d}{dx}(\sin 2x) = 2\cos 2x$, we plot $y = \sin 2x$ along with the two lines $y = 2x$ and $y = -2x$ in Figure 4. We can see that the gradients of this new curve can be greater than 1 and lie between -2 and 2 as expected.

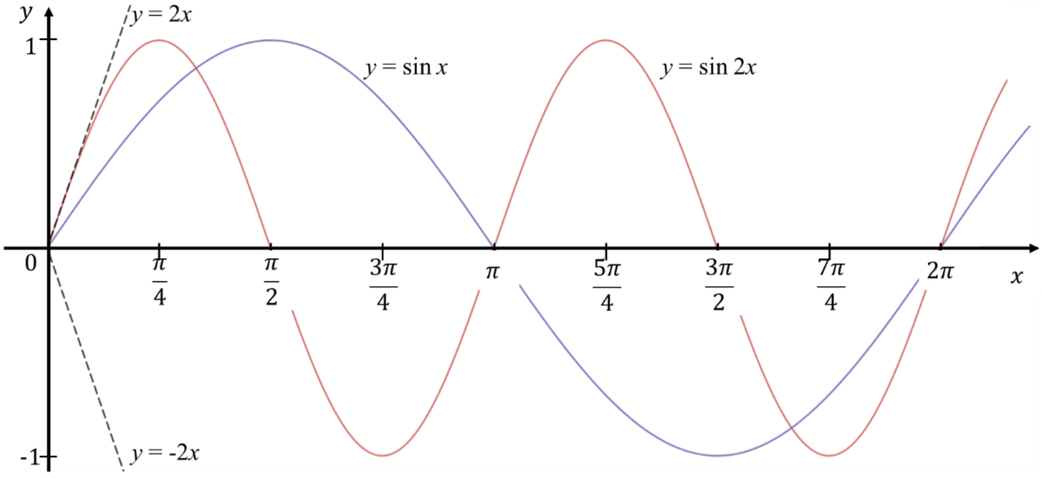

Figure 4. Diagram of $y = 2x$, $y = -2x$, $y = \sin x$ and $y = \sin 2x$

A discursion into the idea of 'unit'

It may be useful here to go into a discursion of what 'units' look like as 'variables'. You may safely skip this section if you wish only to follow the main topic.

What does it mean when we sketch a graph of "$y = \sin x$, where x is in degrees"? Do we number the x-axis as in Figure 5a or as in Figure 5b? Actually, there is no difference but Figure 5b makes things clearer.

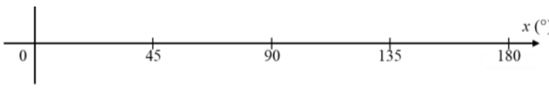

Figure 5a. Graph with the x-axis in degrees (°)

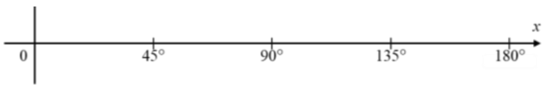

Figure 5b. Graph with the x-axis in degrees (°)

We explore the idea of units in graphs via the following questions:

- What does $y = \sin x$ look like?
- What does $y = x$ look like?
- What does $y = \sin x°$ look like?

The question "What does $y = \sin x$ look like?" is important to understand the limitations of units in graphs. Suppose we have a graph with x in centimeters. This type of graphs is useful to depict results of science experiments. The x-axis will normally look like Figure 6a. But we know that it can be equivalently portrayed as in Figure 6b.

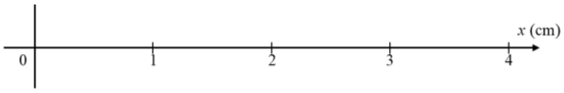

Figure 6a. Graph with the *x*-axis in cm

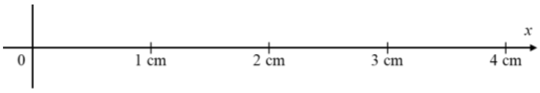

Figure 6b. Graph with the *x*-axis in cm

We can now see that *y* = sin *x* is no longer well-defined for a graph with *x* in cm since sin (2cm), for example, is meaningless. We are reminded that *x* has to be dimensionless for sin *x* to be evaluated. We are also reminded that angle in degrees (°) is a dimensionless 'unit' and that it is actually a scaling of $\frac{\pi}{180}$, that is $x° = \frac{\pi}{180}x$. Thus, *y* = sin *x* is meaningful for a graph with *x* in degrees (see Figure 7a) but will look different from *y* = sin *x* in a graph with *x* as a pure number (see Figure 7b).

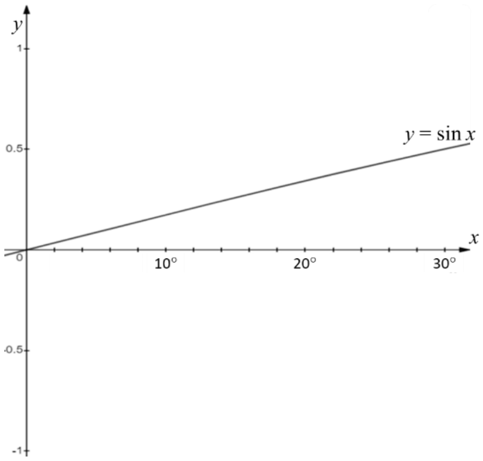

Figure 7a. Graph of *y* = sin *x* with *x* in degrees

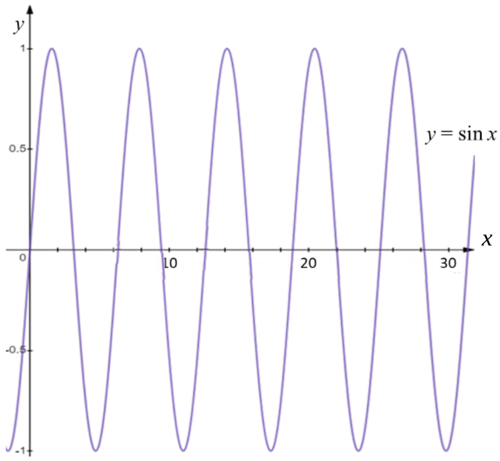

Figure 7b. Graph of *y* = sin *x* with *x* a number

Next, we intend to insert the line *y* = *x* into each of the graphs in Figure 7. We realise that we need to know the units of *y* as well! Since it is usual to have y as a pure number, we will use this unit for *y*. In Figure 8(a), we will have *x* in degrees and in Figure 8(b), *x* will be a pure number.

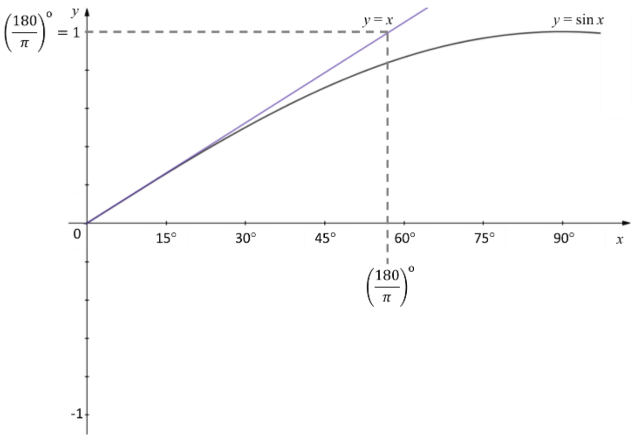

Figure 8a. Graph of $y = x$ and $y = \sin x$ with x in degrees

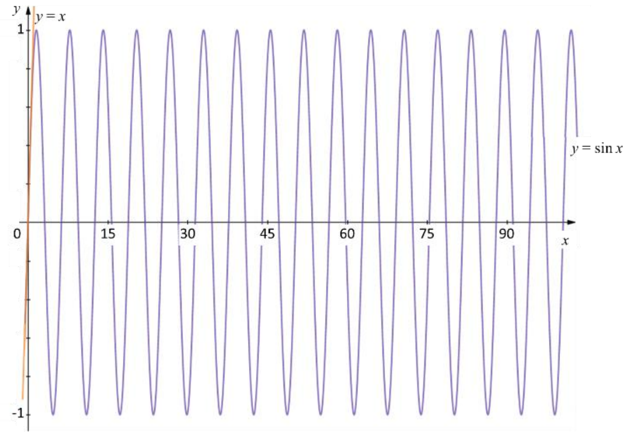

Figure 8b. Graph of $y = x$ and $y = \sin x$ with x a number

By now, we should realise that units affect the shape of the graphs. But do they affect the gradients of tangents to the curve? Some will be surprised to find out that the answer is “No”. In fact, this can be easily checked by observing that the line $y = x$ in each of the four graphs is tangent to the curve $y = \sin x$ at $x = 0$. The reader may want to check for another value of x, say 30° or $\frac{\pi}{6}$ depending on the unit of x, the gradient of the tangent at x for the 4 graphs. This observation reinforces the veracity of the proposition that

$$\text{“}\frac{d}{dx}(\sin x) = \cos x, \text{ no matter what the unit of } x\text{”.}$$

Next, we consider how the curve $y = \sin x°$ compared to $y = \sin x$ would look in a graph with x in degrees (see Figure 9a) and in a graph with x as a pure number (see Figure 9b).

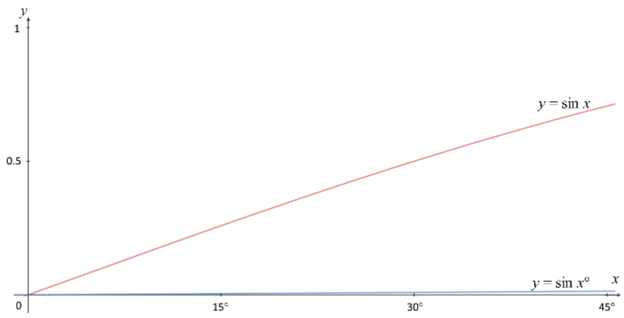

Figure 9a. $y = \sin x$ and $y = \sin x°$ where x in degrees

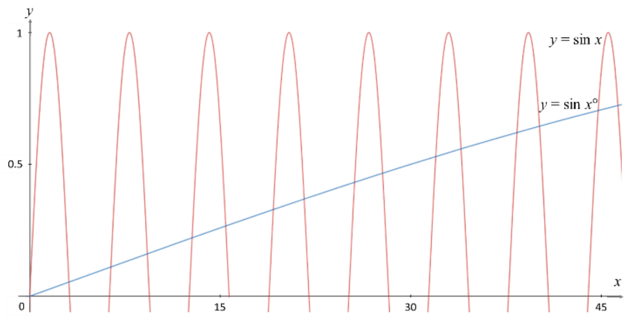

Figure 9b. $y = \sin x$ and $y = \sin x°$ where x a number

Figure 9(b) is the same as Figure 3, where we made the observation that the curve $y = \sin x°$ is much gentler compared to the curve $y = \sin x$. To draw Figure 9(a), we must make sense of what $\sin x°$ is when x is already in degrees. Pedagogically, we utilize an example from a more familiar scenario as follows. The area A of a rectangle of sides x and y, where x and y are in cm, is xy. Thus, if x = 6 cm and y = 1 cm, A = 6cm × 1 cm = 6 cm cm = 6 cm^2. Returning to our work, if x = 30°, then x° = (30°)° = $(30 \times \frac{\pi}{180})° = 30 \times \frac{\pi}{180} \times \frac{\pi}{180}$. Thus, in Figure 9(a), we have exactly the same scaling 'down' by $\frac{\pi}{180}$ of the curve $y = \sin x$ to the much gentler curve $y = \sin x°$. As in Section 3, we would make the same conclusion for the graph in Figure 9(a) that $\frac{d}{dx}(\sin x°) = \frac{d}{dx°}(\sin x°) \cdot \frac{dx°}{dx} = \frac{\pi}{180}\cos x°$. We also come to the analogous conclusion that

"$\frac{\mathrm{d}}{\mathrm{dx}}(\sin \mathrm{x}°) = \frac{\pi}{180}\cos \mathrm{x}°$, no matter what the unit of x".

This discursion will bring the newcomer's use of the word 'unit' closer to that of the old-timer's use of the word 'variable'. While the old-timer teacher agrees to adapt her discourse to the newcomer, the newcomer learner himself must agree to "climb the discursive social ladder" (Sfard, 2008, p. 282) to progress. After seeing how 'units' work, the newcomer should gradually learn to work with the more commonly used 'variable'.

The first principle of differentiation

In any case of doubt, mathematics practitioners should return to the first principle to resolve any fallacy, although we acknowledge that beginning students' difficulty in deriving from the first principle is often compounded by the necessary sophisticated algebraic manipulations (Lim, 2008). In Toh (2008), we have a standard derivation of $\frac{d}{dx}(\sin x)$ using first principles (p. 73). In this section, a parallel derivation is obtained for $\frac{d}{dx}(\sin x°)$.

$$\frac{d}{dx}(\sin x°) = \lim_{h\to 0}\frac{\sin(x+h)° - \sin x°}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)°\sin\left(\frac{h}{2}\right)°}{h} \quad (1)$$

$$= \lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)°\sin\left(\frac{h}{2}\right)°}{\frac{h}{2}}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)° \lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)°}{\frac{h}{2}} \quad (2)$$

Note that (1) is obtained by using Factor Formula for the subtraction of two sines, which is true for both the angle measured in radians or degrees. In (2), the problem reduces to evaluating $\lim_{x\to 0}\frac{\sin x°}{x}$. Any elementary calculus book would show that if x is measured in radians, then $\lim_{x\to 0}\frac{\sin x}{x} = 1$ (for example, Lee, 1993, p. 18). At this juncture, we sometimes encounter Fallacy 3 as stated below.

Fallacy 3

Note that the derivation of $\frac{d}{dx}(\sin x)$ *depends on* $\lim_{x\to 0}\frac{\sin x}{x} = 1$. *The value (1) on the right hand side remains the same as long as values of "x" in* $\sin x$ *and "x" in the 'denominator' are in the same unit. In particular, if the x is replaced by* x^o, *we also have* $\lim_{x\to 0}\frac{\sin x°}{x} = 1$.

As before, we note the incomplete transfer of 'units' from the original equation $\lim_{x\to 0}\frac{\sin x}{x}=1$ to the equation $\lim_{x\to 0}\frac{\sin x^\circ}{x}=1$. If the newcomer has by this time agreed to be more careful about the use of units, he or she would easily realise his or her own mistake in stating the fallacy.

In addition, it may be instructive for the learner to see that $\lim_{x\to 0}\frac{\sin x^\circ}{x}\neq 1$ in a more direct way. We do this by repeating the same argument in Lee (1993, p. 18) when the angle *x* is replaced by x°. By comparing the arc length of a unit circle with subtended angle x° and the corresponding right-angled triangle with one angle x° and one radius of the unit circle as one side, the ratio of the length of the side *AB* of the triangle to the arc length *AC* (Figure 10) is computed below:

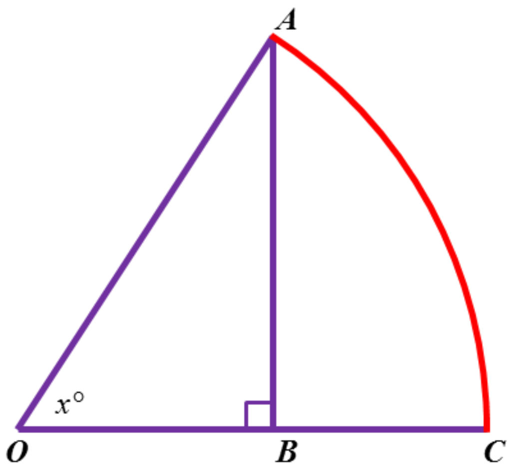

Figure 10. Comparing the arc length *AC* and the length *AB*.

In this case, arc length *AC* is $\frac{x^\circ}{360^\circ}2\pi=\frac{x\pi}{180}$ (if we "cancel" the unit of angle measurement from the numerator and denominator) or $\frac{x^\circ}{2\pi}2\pi=x^\circ$ (if we express the denominator in radians). Continuing along these two paths and comparing the arc length *AC* and the length *AB*, as $x \to 0$, we have that $\frac{\sin(x^\circ)}{x\pi/180}\to 1$ or $\frac{\sin(x^\circ)}{x^\circ}\to 1$. We conclude that the statements $\lim_{x\to 0}\frac{\sin x^\circ}{x}=\frac{\pi}{180}$ and $\lim_{x\to 0}\frac{\sin x^\circ}{x^\circ}=1$ are both correct. However, in deriving the formula for $\frac{d}{dx}(\sin x^\circ)$, the limit that is required (see (2)) is $\lim_{x\to 0}\frac{\sin x^\circ}{x}$, which is $\frac{\pi}{180}$.

Conclusion

Mathematicians have found abstraction of mathematics beautiful. Through abstraction, mathematical principles can apply to a wide variety of problems independent of the physical context. While contextualizing mathematics helps the learners appreciate the application of mathematics, returning to first principles helps one resolve any uncertainties in particular real-world contexts. Mathematician educators are deeply aware of the moorings of their pedagogical ship to the mathematical port. Between a 'physical' explanation that relies on less well-defined words such as 'units' and an abstract concept rich in history and well-defined words such as 'variables', the mathematician educator would choose the latter. She would then "climb down" her discourse as much as possible to meet the discourse of the serious learner. Indeed, in our discussion above, the fallacies resulted from a discourse that employed the word 'units' in a non-rigorous manner.

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