The Mathematician Educator
2021, Vol 2, No. 1, 33-46

A Listing Approach for Counting Problems

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It is well-known that Counting Problems are difficult for many students. Mistakes such as the wrong use of formulas or the insensitivity to over/under-counting are common. This study draws on the work of Lockwood (2013, 2014) to conceptualise the interacting components in the work of solving counting problems. In particular, we implemented a "listing approach" in the teaching of counting problems in a Polytechnic course for engineering students in Singapore. From the close interview of three students in the course that corresponded to three profile types, we evaluate the specific usefulness of the approach for each of these types of students. We also propose a provisional model that can guide the restructuring of a course on combinatorics.

Keywords: counting, problems, problem-solving, listing approach, visualise, generalise.

Introduction

Because memorise-and-apply is such an efficient way to solve the many textbook-type 'problems' encountered in one's school life, it is not easy to persuade students to move away from this mode in situations where genuine mathematical problems – those that do not yield direct solution strategies at first glance – are presented to them. As mathematics problem solving is at the heart of the Singapore mathematics curriculum (MOE, 2020), mathematics teachers have an interest in helping students learn how to tackle unfamiliar problems.

One type of problems that can potentially help students shift towards a problem-solving mode of thinking is "counting problems". We think the nature of these problems lends well to the objective of teaching problem solving for these reasons: (a) counting problems have low content requirements – only basic knowledge in numbers, multiplication, and addition – and so the focus for students can rightly be on the problem solving process rather than in acquisition of new mathematics content; (b) the numerous types of counting problems would reveal the ineptness of simplistic formula-application, thus providing motivation for approaching these problems in a fundamentally different way; and (c) the use of problem solving approaches can quite quickly lead to understanding of the problem structure and hence enhance buy-in to this approach. The research reported in this paper relates to the last point. In particular, we explore the use of a listing approach in a sub-module on counting problems.

Problem solving

Since the seminal work by Pólya (1945) on problem solving, there has been a lot of research in this area. It is not our intention in this short article to review the many decades of mathematics problem solving research. We note a somewhat revival of this area of research in Singapore over the last decade. The emphasis, a la Schoenfeld (2007), of this wave of research is less on theoretical foundations and more on how to translate decades of theory building into workable and sustainable problem-solving instruction in Singapore secondary classrooms (e.g., Toh et al., 2011; Leong et al., in press).

For the purpose of this study, our focus is on one aspect of teaching problem solving – helping students develop a *problem solving disposition*, which was described as: "when confronted with an unfamiliar mathematics question, instead of giving up early, [the student] will try ways to tackle the problem productively; and instead of relying on the teacher for every step, that student will own the problem by way of checking and going beyond the initial incorrect strategies" (Leong et al. 2013, p. 1257).

Clearly, such a disposition is not natural for students who are more used to a mere formula-application view of mathematics learning. As such, to help students develop such a disposition, there is a need for careful design of instruction, including the deliberate selection of problems that would lend themselves to a demonstration of success in such a disposition. As mentioned in the introduction, we think that counting problems are suitable for such a purpose. But apart from the selection of problems, we need to have an accompanying theory of action in using such problems in an instructional setting. We thus turn to literature that are more specifically on counting problems.

Solving counting problems

Combinatorics is a topic where teachers encounter many types of student errors. Batanero et al. (1997), based on their analysis of the responses of 720 fourteen- and fifteen-year-olds, listed 14 types of errors. The more common ones – and the ones more commonly cited, anecdotally – include: order (confusing the criteria of combinations and arrangements), objects (confusing distinct objects and identical objects), repetition (failure to consider repetition of objects is allowable), operations (confusing the criteria for Addition Principle and Multiplication Principle), partition (that results in over-counting or under-counting), and formula (applying a wrong formula).

When thinking of a way to instruct as to minimise these errors, instead of addressing these errors separately, there is a need to examine the more fundamental origins of these errors. We think it has largely to do with how students experience these problems. If they are merely presented with categories of counting problems with set techniques corresponding to each of these categories, then it is no wonder that a fixed formula-application mindset – and hence error in not scrutinising the criteria for their application – develops. Thus, in learning the solution to these problems, students should be given the opportunity to find out for themselves how and why combinatorial distinctions (such as combination versus arrangement, distinct objects versus identical objects) are important. This is where problem solving provides the space for students to carry out such explorations.

More specific to counting problems, Lockwood (2014) proposed a "set-oriented perspective". By this, she meant that the emphasis when solving problems should be shifted to identifying the sets of outcomes arising from the counting process. She provided the example of the Password Problem: A password consists of eight upper case-letters. How many such 8-letter passwords contain at least three E's? She pointed out that a common erroneous solution provided by students were: $\binom{8}{3} \times 26^5$. A set-oriented perspective would mean that students would focus on the outcomes being counted – such as, in the case of EEEABEEE, it can be EEE_ _ _ _ _ or _ _ _ _ _ EEE in the $\binom{8}{3}$ stage, which can lead to the same outcome in the 26^5 stage – and thus accentuate the likelihood of 'catching' his own over-counting. She added that, "Without considering outcomes, it can be very difficult to determine why such a seemingly logical counting process is incorrect" (p. 32). A similar argument was forwarded with respect to the error of "order". "Deciding whether order matters can be difficult for students, and it easily devolves into relying on memorized formulas or vague intuition. Although students do not always think of it this way, the commonly used expression of "order mattering" heavily relies on the nature of the outcomes being counted" (p. 33).

Listing as a strategy to generate the sets of outcomes

This set-oriented perspective was further developed as Lockwood and Gibson (2016) studied students' use of "listing" as a strategy to generate the sets of outcomes required by counting problems. That systematic listing is a problem-solving strategy traces all the way back to Pólya (1945). In combinatorics, it is more specifically tied to the process in which students concretely enumerate and visualise the sets of outcomes. In this context, English (1991) identified the Odometer Strategy that young children used, where the listing procedure is one which was cyclical with some pivots kept at constant in each count, and systematically changed for the next count and so on.

Lockwood and Gibson (2016) focused on how undergraduate students demonstrated productive and non-productive listing strategies. It was found that the more successful counters were the ones who used listing. Also, in their qualitative analysis, they noted a surprising result: a student who was successful through listing in one problem went back to an earlier problem and was able, through listing, to correct the errors she made in that problem. This meant that the student's experience in productive listing was not confined to one problem; the experience gave her the motivation to use it as a useful strategy for other counting problems. Moreover, it was found that even partial lists can be productive. In fact, of all the correct answers produced by the students in the study, 63.2% used partial lists. In other words, for these students, listing was not a strategy for complete enumeration of the outcomes; it allowed them to concretise the outcomes for the purpose of generalising the sets for the purpose of counting them correctly. They concluded that "[s]ystematic listing helps to accomplish [the] aim by orienting students with what constitutes a desirable outcome, ensuring that they understand what they are trying to count. … [O]ur findings provide direct evidence that listing outcomes is one practical, concrete way in which outcomes can be made the focus of the activity of counting" (p. 268).

In our study, we adopted some of the ideas arising from the literature that is reviewed in the preceding sections. The first author (subsequently referred to in the first person singular), as a lecturer in a Polytechnic in Singapore, is involved in the teaching of combinatorics in some mathematics courses for Engineering students. I can resonate with the errors that students make which were identified in the literature. As such, I am deeply interested in how some of the research-based theoretical insights can help improve the teaching of combinatorics in these courses. However, as counting problems form a very small proportion of coverage in these courses and the instructional time allocation for combinatorics is very limited, it is not realistic to envisage a full problem-solving approach for this topic. We thus conceived of the use of the listing strategy as an augment to the teaching of formulas for the topic. In particular, this paper is a report of an exploratory study into how some polytechnic students respond to the listing approach to counting problems within an instructional context where straightforward application-of-formula solutions are available.

Context

The topic of Combinatorics was part of the Probability and Statistics module taught over 15 weeks to second year Engineering students at Nanyang Polytechnic. The coverage for Combinatorics was allocated only 2 weeks out of the entire duration. Students were to view the e-lectures on their own followed by a 2-hour tutorial session on a weekly basis. In view of COVID-19 situation, the tutorial sessions were conducted over Zoom sessions by me.

The e-lectures comprised 3 video recordings done by the moderator of the course – a colleague of mine who was not part of the project team. In these e-lectures, he covered the Addition Principle and the Multiplication Principle, Permutation and its notation nPr, Combination and its notation nCr, and the application of these ideas to solve some standard counting problems. The tutorial questions were set by the moderator and my responsibility as a tutor of this course was to follow up the contents of the e-lectures by demonstrating how these ideas can be applied in the solution to these tutorial problems.

Given the limited time of only two 2-hr tutorial sessions over Zoom, the approach I took was to exemplify the use of the listing strategy judiciously: where the problem in the set of tutorial questions fitted into more standard types, I demonstrated the application of formula; but I slowed down at those other problems which would lend themselves to the listing strategy as a way to do problem solving. Because of this trade-off, I was not able to cover all the questions in the tutorial sheet. An example of a problem that we considered standard is:

> Using the letters from the word *COMPUTER*, find (i) the number of words that can be formed using all the letters, (ii) the number of 4-letter words that can be formed.

An example of a problem that we judged to be suitable for the problem-solving approach was:

> In how many distinguishable ways can the letters in the word PAPAYA be arranged?

To the trained observer, this problem requires no more than a 'standard' one liner of 6!/(2! x 3!) – a straightforward application of formula. But, in line with the purpose of this study, we look for problems that lend themselves to the listing approach as a counting process towards the solution of these problems. We identified the above problem as one such candidate. But so as not to fixate the students with a mere formula-application route, I first presented solutions using the listing approach before showing them the direct formula application.

For the rest of this section, we provide a brief description of how the problem was discussed during the first tutorial session for the reader's reference.

A student identified that there are 6 letters in the word with 'PAPAYA' with 2 identical 'P' and 3 identical 'A'. For the first approach to solving the question, I drew 6 blanks and asked students to consider how many ways can the 3 identical 'A' be placed on any of the 6 blanks. A student responded that it is 120 ways by using 6P3 not realising that the 'A's are identical. I took the opportunity to clarify with the class the difference between nPr and nCr. Hence, it is 6C3 =20 ways instead. By first fixing the positions of the 'A's, I can pivot – consistent with the Odometer Strategy – to list the remaining 3 positions of the 2'P's and 1 'Y' to generalise the "sets of outcomes" (Lockwood, 2013). Thus, using the multiplication principle, for 20 different arrangements of the 3 'A's, there are 20 x 3 = 60 ways of arranging the word 'PAPAYA'. Figure 1 shows the workings I provided through the Zoom whiteboard feature.

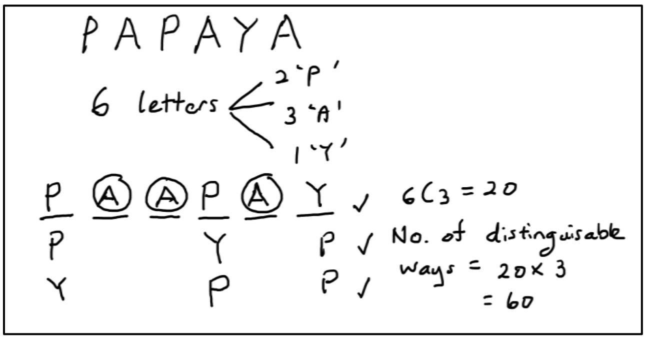

Figure 1: A re-written extract of the solution to the problem using the first approach

To show students that there is more than one way to solve the problem, I illustrated a second approach. I drew 6 blanks in a row and placed a 'P' in the first position. There are 5 remaining blanks to fill the second 'P'. Next, moving the first 'P' to the second position, there are 4 remaining blanks to fill the second 'P' after the first 'P'. This process is repeated until the second 'P' has only one remaining blank to fill after the first 'P'. Using the Addition Principle of Counting since the events are non-overlapping, the number of ways to arrange the 2 'P's is 5+4+3+2+1=15 ways or 6C2 ways. This really is a listing approach to show the various position to place the 'P's and an alternative to the formula 6C2. This is in line with Lockwood's (2013) model of linking the sets of outcomes (the possible position of the 'P's) from the counting process (that is, by systematic listing and the addition principle) which results in the same answer as the formula (6C2). Considering a particular arrangement of the 2 'P's, systematically listing all the possible arrangements of the remaining 3 'A's and 1 'Y' would give us a total of 4 ways. Hence, for 15 different arrangements of the 2'P's, there are a total of 15 x 4 = 60 ways of arranging the letters of the word 'PAPAYA'. Figure 2 shows the workings of this alternative solution.

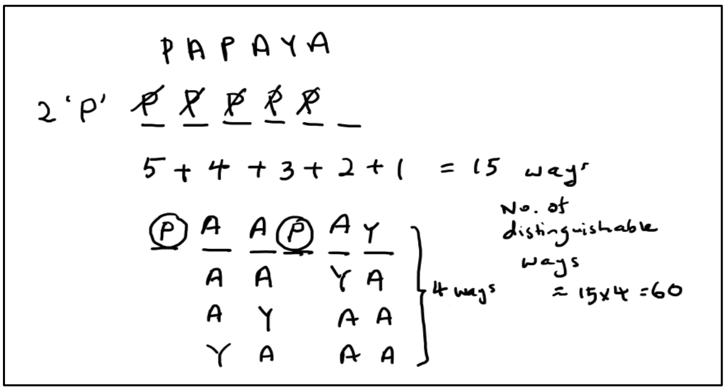

Figure 2: A re-written extract of the solution to the problem using the second approach

Consistent with the requirement of this study, we need to show the students the direct formula method. I thus illustrated a third approach to solve this problem using the formula n!/(k1!x k2! x …..km!), where n is the total number of objects, k1 is the number of identical type 1 objects, k2 is the number of identical type 2 objects and so forth. Since there are 6 letters with 2 identical 'P's and 3 identical 'A's, the number of ways to arrange the word 'PAPAYA', using the formula, will be 6!/(2! x 3!) = 60 ways.

Method

Since the purpose of this study is to investigate how students would respond to the listing approach that were taught to them over the two tutorial sessions, we wanted to find out whether and how some students use the listing approach after it had been taught to them. So, towards the end of the first tutorial session, the students were given this problem taken from the tutorial sheet to work on for 10 minutes:

> Janet has 10 different books that she is going to put on her bookshelf. Of these, 4 are Chemistry books, 3 are Biology books, 2 are Statistics books, and 1 Physics book. Janet wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?"

The selection of this problem is in line with its potential to bring out the listing strategies that would help the students in producing the cases which they may generalise to obtain the desired sets of outcomes. The students were asked to work on this problem independently. They were to submit their written solutions to me – via email through scanning their written solutions – at the end of the session.

48 out of 52 students submitted their solutions. 26 students managed to get the correct answer by using formula. Theirs was a one-liner answer of 4! × 4! × 3! × 2! or 6912. 7 students obtained the correct answers without using formula; and 15 students did not get the correct answer. For the purpose of this study, we were less interested in correct answers than in their response to the listing approach. But their solutions provided us with some information with which to choose the candidates for the next stage of data collection, which was to interview some students to find out if the listing approach had an effect in how they engage with counting problems. We identified three candidates: *Kok Keong* who demonstrated productive partial

listing in his solution, ***Haojie*** who attempted to be systematic in the partitioning of the sets of outcomes but had errors conceptually and also in the calculation of the cardinality of the sets, and ***Minah*** who attempted to apply a formula wrongly to obtain the answer.

But prior to interviewing them, since we wanted to find out if the listing approach was still seen to be useful to them when a more straightforward formula was applicable, I worked through the solution to the question with the whole class in the next tutorial session which was a week later. I highlighted that the books of the same subject are different since the question did mention that Janet has 10 different books. Furthermore, Janet is an individual and we do not expect an individual to be having identical books of the same subject. The next step is to group the books of the same subject together. I drew a rectangular box to contain the 4 Chemistry books, naming them as C1, C2, C3 and C4. Another rectangular box to contain the 3 Biology books, naming them as B1, B2 and B3. The 2 Statistics books (S1 and S2) into the third rectangular box and another box to contain the Physics book. The 4 boxes were drawn in a row. The number of ways to arrange the 4 boxes is 4! = 24 ways keeping the arrangement of the books within each box unchanged. For one arrangement of the boxes, there are 4! = 24 ways to arrange the 4 Chemistry books within the box, 3! = 6 ways to arrange the Biology books within the box and 2! = 2 ways to arrange the Statistics books within the box. There is only 1 way for the Physics book to be in the box since there is only 1 Physics book. Hence, the total number of ways for Janet to arrange her books into her bookshelf is, by Multiplication Principle, 24 x 24 x 6 x 2 x 1 = 6912. Figure 3 shows the diagram I presented on the Zoom Whiteboard share-screen as shown to the students.

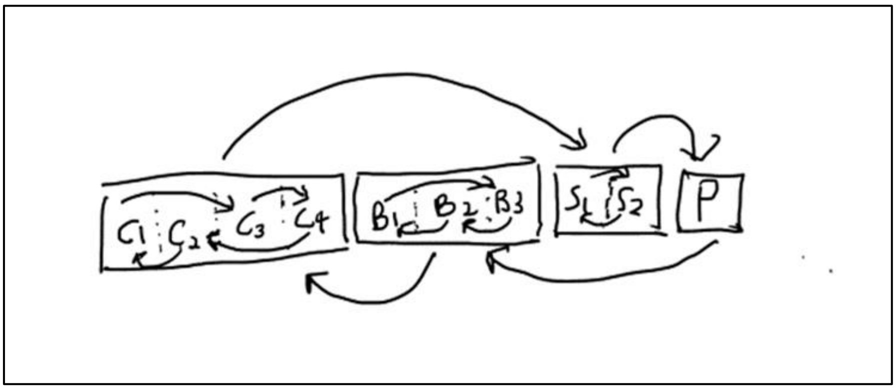

Figure 3: A re-written extract of tutor's solution to the problem

The first thing I did during the interviews was to present on Zoom screen their own solutions of the problem – the one they submitted to me at the end of the first tutorial session. I then asked them to explain to me their solutions in light of the additional inputs they gathered from me in the second tutorial session where I discussed its solution (as described in the preceding paragraphs). The purpose of doing this was to listen to the student's thinking process behind their original solution and possibly new insights after they had seen my solution. It was also to investigate if the use of the listing method had had any influence on how they now approach the same problem.

After asking them to explain their solutions, I followed up with slightly different paths of interview for each of the interviewees. For Kok Keong who had demonstrated the use of productive partial listing in his original solution, the interview was an opportunity to drill deeper into the specifics of his approach – how the listing was helpful towards the obtaining of the required sets of outcomes, how he determined when to stop the listing so that he could fill in the rest of the sets of outcomes productively without actually listing them in detail. For Haojie who attempted to be systematic but made mistakes, the interview provided an opportunity for him to review his misconceptions. Where relevant, I also presented the listing approach as a way to uncover these misconceptions – with a view of examining his response at that stage to the listing method. For Minah, since her solution was formula-based and incorrect, it was not useful from this study's perspective to drill deep into her thinking behind the use of the formula; instead, the interview was geared towards presenting her again with the listing-based solution with the view of studying her response to this approach.

Each interview was analysed separately by partitioning it into natural episodes (such as: an episode of Student X explaining the reason for using the Addition Principle instead of the Multiplication Principle) and then putting them together chronologically as a coherent narrative. In the sections that follow, we present the chronology of events during the interview as a context for the main focus of this study. As mentioned earlier, the focus is on each student's response to the listing approach, and where relevant, the development of their approaches.

Student Kok Keong

Kok Keong was interviewed as we found him a 'standout' candidate in terms of exemplification of the use of a problem-solving approach to the counting problem; and more specifically, in his use of productive partial listing as a strategy towards obtaining the sets of outcomes. Figure 4 is an extract of his written solution.

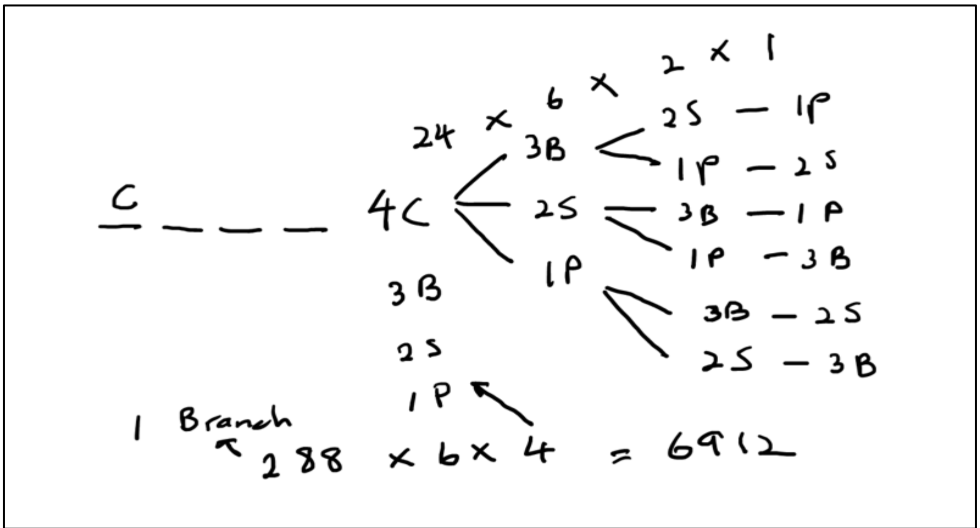

Figure 4: A re-written extract of Kok Keong's solution to the problem

During the first part of the interview, Kok Keong explained his written solution as shown in Figure 4. He began from the left side of the diagram where he saw the arrangement as comprising four types of books (hence the four blanks) and he started the listing with the four Chemistry books being placed first from the left of the shelf. The "tree diagram" (as he called it) was his way of systematically listing the various sub-cases under this case. This level of listing on paper was sufficient for him – as he could see that to obtain the number of outcomes for each sub-case, he needed only to use the formula for permutation to obtain the 24 (which is

4!), 6 (3!), 2, and 1, and to take the product of these numbers. He also did not have to perform the same counting for the rest of the "branches" (his language) as he could see that the number of outcomes are the same (that is, 288) though the order of the factors in the product are different. Up to this point, it was clear to us that Kok Keong had an organised plan to frame his listing and he exercised carefulness in systematically listing all the sub-cases so that all outcomes are counted.

Subsequently, he went back to the case of which type of books to arrange first from the left on the shelf and noted that other types of books can be arranged first instead of the four Chemistry books. In other words, his overall approach was to keep the constant "pivot" of type of books to arrange on the left and then proceeded to count the outcomes before returning to change the pivot accordingly – that is, he was using the "Odometer strategy" of listing (English, 1991). He did not have to redo the complete "tree diagram" listing of these other cases as he could see that the number of outcomes is the same as the first case (that is, 288 × 6). Since there were 4 cases in total, he could proceed to take the product as shown in Figure 4 to be the required number of outcomes for the problem. On the whole, Kok Keong's approach to this problem exemplified a number of characteristics of efficient use of productive partial listing as reviewed earlier in this paper. He also employed a strategic combination of the use of counting processes with judicious application of 'formula'. He was confident of his approach – he said "it is not so complicated" – and was able to articulate it clearly during the interview. The fact that he was able to readily recall and explicate the method even after more than a week since it was last discussed showed the robustness of his grasp of the solution to the problem.

When asked about the origin of this way of approaching the problem, he cited "common sense" and "previous learning". He also mentioned that it was "in the syllabus" of the course. Kok Keong did not specify the links between the method advocated by me and his method for the problem. But a cursory comparison of Figure 1 and Figure 4 shows similarities especially in the use of spaces as a notation to indicate positioning of permuting objects. Thus, we think it reasonable to assert that the manner in which I taught the listing method at least legitimised his approach. Perhaps, it even reinforced his resolve to use problem solving as a viable way to learn mathematics.

He also made reference to this approach in comparison to the one subsequently discussed by me (see Figure 3). He acknowledged that his original method was more "tedious" compared to the use of formula for 'set' problems, but it helped him "understand the problem". He mentioned that he welcomed multiple solution approaches to problems as it was "fun" for him to learn different methods. In fact, upon reflection, he saw the two methods as sharing "the same idea" – his original method was an "expansion" of the "one-liner" solution presented by me. When asked if he will persist with using this problem-solving approach in future (even after seeing more 'efficient' solutions), he said that if he sees a similar problem where formula application is straightforward to him he will indeed do so; but if he cannot remember the links between problem and formula, he will again use this problem-solving approach to make sense of the problem. In other words, the problem-solving approach via listing is a default fall back method for him for unfamiliar problems, and from it he might be able even to recover the logic of the formulas.

Student Haojie

Haojie was interviewed as we found that he attempted to approach the problem using the blanks-to-fill (as taught to the class in my tutorial sessions with them) although he did not manage to arrive at the correct answer. Figure 5 shows the solution he submitted.

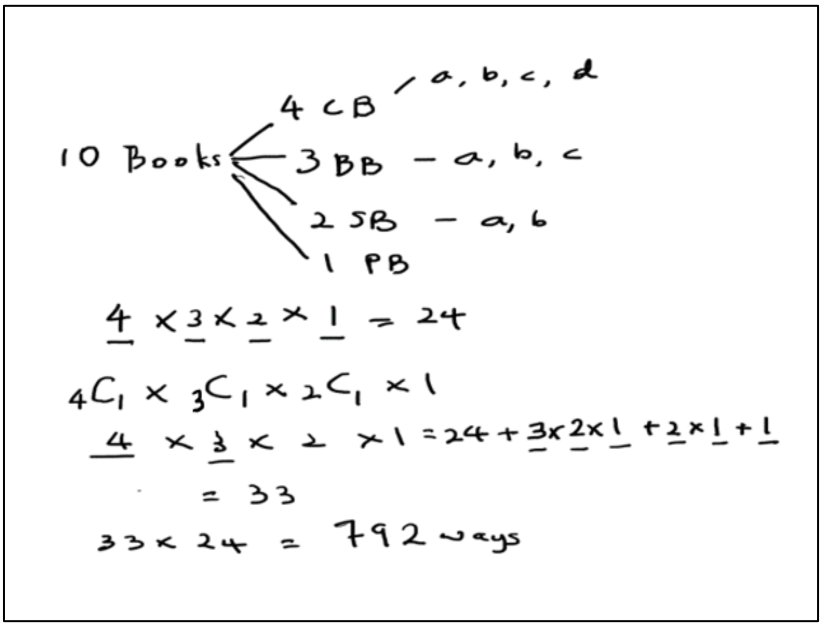

Figure 5: A re-written extract of the solution submitted by Haojie

The first part of his written work was his way of translating the information in the problem into a visual form – “4 CB” stands for “4 Chemistry books” etc. Next, he took 4 x 3 x 2 x 1 = 24 ways to represent the number of arrangements where books of the same subject are treated as one block. I asked him how he arrived at this answer. He struggled a bit – at one point pausing for 10 seconds – but subsequently explained that “since the Chemistry books cannot be mixed together with the other subjects then to me the order can be 4 Chemistry books [first] or another order can be like 1 Physics book, 3 Biology books then 2 Statistics books”. It seemed that he was justifying the formula (of 4!) by mental listing – in this case, verbalised in order to convince me and possibly himself – to visualise the set of outcomes when arranging these types of books. This conjecture was further strengthened later in the interview when he came to the next line of his working: He realised himself that his 4C1 x 3C1 x 2C1 x 1 step was wrong. When asked how he arrived at 4C1, he mentioned that it should instead be 4P4 and went on to do a similar justification by saying, “because there are 4 books and any of the books can be the first one and any of the books can be the second one and so on”. He used the same reasoning to justify 3P3 for the number of ways of permutating the Biology books and subsequently generalised the same for the Statistics and Physics books. In both the cases of justifying for 4! (as the number of ways of arranging the types of books) and nPn (as the number of ways of arranging n different books), he instinctively used formulas that were taught to him but also fell back on (mental) listing to strengthen his claims.

On his next working step “24+3x2x1+2x1+1”, I further asked if he was able to explain addition instead of multiplication of the numbers. Here, he was unable to use listing – perhaps it became too complicated to do so mentally – to explain. Thus, I proceeded to demonstrate using systematic listing to show that the Multiplication Principle should be used instead. I used the

Zoom Whiteboard feature to show C1C2C3C4B1B2B3S1S2P as one way of arranging the books according to the criteria in the problem. I explained that for each way of arranging the Chemistry books (say C1C2C3C4), there are (3!2!) ways of arranging the rest, and so since there are 4! ways to arrange the Chemistry books, the relationship between 4! and (3!2!) should be multiplication. Halfway in my explanation, he interjected to say, “I got it!” Unlike Kok Keong who was proficient with the productive partial listing approach in navigating all the hurdles, Haojie’s use of a form of simple mental listing did not go quite as far. Nevertheless, he was able to use a limited form of listing to serve as fallback for some formulas he was familiar with.

When asked how he would normally approach a question, he said that he would try to identify which formula could possibly fit in to the question and just apply the formula. Haojie expressed that being taught different approaches to solving a question “is quite good”. He said that knowing the different ways of solving a question would mean that he can now have more options instead of just trying to identify which formula to use in solving the question.

Student Minah

Minah was interviewed as we would like to find out how she responded to the listing approach as a way to help her see her mistakes in her original solution. Figure 6 shows the solution she submitted.

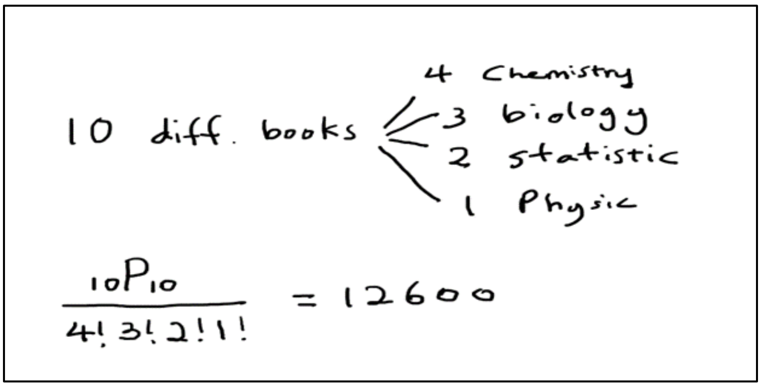

Figure 6: A re-written extract of the solution submitted by Minah

Minah explained that she first tried to categorise the 10 books into 4 Chemistry books, 3 Biology books, 2 Statistics books and 1 Physics book. She said that she originally assumed the books within each category to be identical and then applied the formula nPn/(k1! x k2! x … x km!). It was at the second tutorial where I explained the solution of this problem that she realised that she wrongly assumed identical books within each category. She was quick to add that she still “do not understand” the working towards the final answer of 6912. In fact, throughout the interview, she repeatedly expressed her desire to “understand” the working steps. With respect to this problem, in particular, she raised the confusion between Permutation and Combination. I explained through giving an example of three objects A, B, and C. Through verbal listing, I explained that ABC, BAC, and CAB are different arrangements or permutations; whereas, in the context of selecting a committee comprising A, B, and C, the above three arrangements would be considered one committee combination. She expressed that she understood this explanation.

She was then asked how she would now approach the same problem. She said she would begin with actual labelling of the books as C1, C2, C3, C4, B1, B2, B3, S1, S2, and P and arranging them in exactly this order. She then proposed another arrangement by rearranging the Chemistry books. After a while, with some coaxing, she realised that there were 4! ways of arranging the Chemistry books, 3! ways of arranging the Biology books, 2! ways of arranging the Statistics books, and 1 way of arranging the Physics book. The product of these numbers yields 288 ways. She had difficulty with seeing why the Multiplication Principle was used in this case. I provided an explanation that was similar to the one I gave to Haojie.

She was initially unable to see why 288 was not the answer to the question. I explained through verbal listing of interchanging the positions of the Chemistry and Biology books, that is, when the type of books is seen as a block that can be moved, that new arrangements are formed. She expressed she now understood how these additional outcomes come about and in fact she offered the answer that there were 4! ways of arranging the type of books (that is, 4! permutations of C, B, S, and P). This led to the final answer of $4! \times 288$.

When asked on how she would normally approach a question in the past, she said that she would look for the formula which she thought could fit in and apply the formula straightaway. But she has since realised in the course that it often did not satisfy her quest for "understanding". During the interview, she showed in a number of occasions how the listing approach led to a better understanding of how the answers were obtained. As such, I asked if she would attempt this alternative listing approach in future when a particular formula does not make sense to her. She readily replied that she would certainly do so.

Discussion

First, we noted that the errors displayed by the students reflected some of the categories pointed out by Batenero et al (1997). In particular, Haojie and Minah made "error of order" (in their confusion over permutation and combination) and "error of operations" (in their confusion over the use of Addition Principle and the use of Multiplication Principle). This study shows that these errors were indeed common. In fact, that was the reason for adopting the listing approach to find out if it will help the students overcome the conceptual hurdles undergirding these ideas. As it turned out, Haojie managed to figure out on his own through mental listing the difference between combination and permutation; but he could not tell whether to use the operation of multiplication or addition in combining 4!, 3!, and 2!. But he was able to 'see' that multiplication should be used when I began to demonstrate through actual listing of the outcomes and attempting to count a set of such desired outcomes. Similarly, Minah – though not adept to listing as a habit for herself – was receptive to my demonstration of listing to explain the differences. She was open to this way of helping her 'understand' the solution. The findings show that the use of systematic listing – as advocated by Lockwood and Gibson (2016) – is promising in helping students overcome these errors in counting.

Second, we can go beyond mere assertion that the listing approach was useful for students – to calibrating how it can help students of different profile types. The three students studied provided three sample profiles of students of combinatorics: (a) over-reliance on formula-use as default mode of approach to problems (as shown by Minah); (b) possess an overall strategy for problem solving but lacking in facility to check correctness of steps (as shown by Haojie); (c) uses productive partial listing effectively to obtain the sets of outcomes and to count them using formula application judiciously (as shown by Kok Keong). The study shows that advocating the listing approach benefits all of these situation types as exemplified by the respective students, but perhaps in slightly different ways: for Situation Type (a), the emphasis on listing is to move students away from a rigid application of formula and to show them that listing is a useful way to help them 'understand' the formula and the solution. In the process, they may actually obtain correct answers for counting problems which in turn further persuades them to adopt this approach. We should persuade students that 'understanding' is more productive than mere unthinking formula-application. For Situation Type (b), we can encourage students to go beyond 'mental listing'. In order to better visualise the sets of outcomes, especially in situations that are more complex (such as the one in the Books Problem), it helps them to actually list out on paper. This can become a useful metacognitive facility for students to check their working steps (e.g., whether it should be addition or multiplication); For Situation Type (c), we can further encourage students to be judicious in the use of partial listing by providing alternative solution strategies based on different ways of listing – such as discussed in the Method Section for the PAPAYA Problem.

Third, the current disposition of many students when they come across a problem is direct formula-application (as shown in the upper portion of Figure 7). This has been shown in the case of Minah and in the majority of the 15 students in the class who did not obtain the correct answer. We think that the first step in helping students to move away from this 'default' disposition is to develop a problem-solving disposition (as reviewed in the earlier portion of this paper) that is closer to that which is depicted in the lower portion of Figure 7. This means that when confronted with a genuine counting problem, we hope that students would instead develop a habit to use systematic listing to visualise the sets of outcomes. Indeed, they do not need to enumerate the full set in most cases; they only need to do a partial list in order to mentally generalise to the whole set of desired outcomes. Once they can visualise the sets correctly, they can then proceed to count them – using relevant formulas to do so, if they are familiar with them. Or, they may fall back on the basic principles to do the counting (as was the case of Kok Keong, where he relied on no more than the fundamentals of the Addition Principle and the Multiplication Principle). In other words, we are not advocating an approach where strictly no formulas are taught and where students have to do the counting without them – we think this is unnecessarily extreme. Rather, the approach is that they may use formula if they are familiar with them and use them appropriately. Where they are not, they need not give up; they can still figure out the counting from more basic first principles.

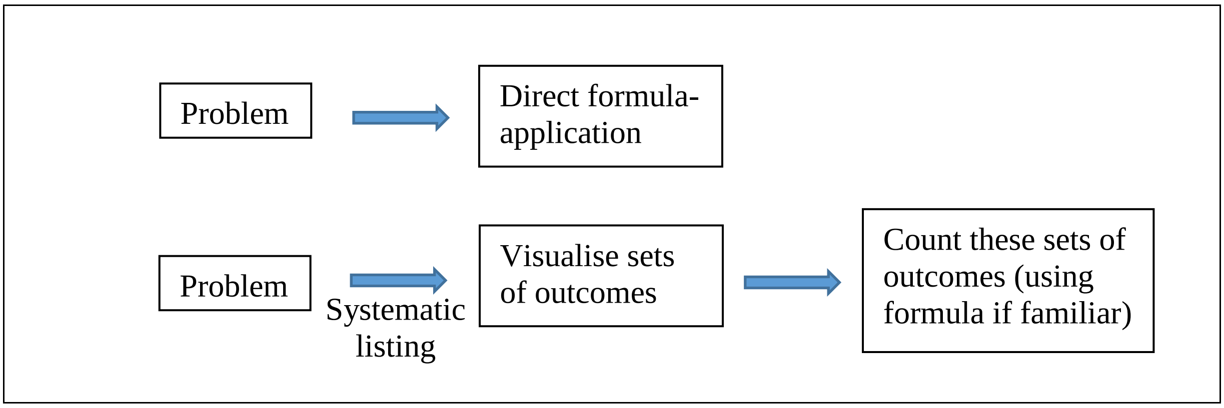

Figure 7: Contrast between a 1-step and a 2-step approach to counting problems

This commitment to this 2-step way of thinking (versus the 1-step direct formula-application) would mean a radical re-working of the relevant curriculum (briefly mentioned under Context) for this topic. For example, instead of frontloading formulas and illustrating each with worked examples (which is the main approach taken in the current curriculum), we can frontload suitable problems – such as the PAPAYA problem – and use the 2-step approach to illustrate how we obtain the sets of outcomes and how to count the cardinality of these sets. We can then inductively introduce the formulas posthoc. Also, if we take this approach, it will mean that more time is needed to illustrate the listing processes; this will then mean that some content aspects of the current course that does not support this approach nor are helpful for other components of this course should be removed. This can be a part of a follow-up study to this current exploratory study and it is a project in its own right.

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