The Mathematician Educator
2022, Vol 3, No. 1, 62-74

Teaching and Learning Complex Numbers through Problem Solving

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With reference to complex numbers, it is argued in this paper that attention should not only be focused on the practical usefulness or the aesthetics of mathematics to make mathematics attractive to students. Teachers could ride on the affordance of the problem solving mathematics curriculum framework in engaging students in activities that reveal the "power" of mathematics in solving mathematics problems and generalizing the results. The paper illustrates how portions of complex numbers, a pre-university mathematics topic, could be introduced through the various stages of mathematical problem solving. The use of complex numbers is a natural progression from basic algebraic manipulation at the secondary level and could be introduced through expanding a problem in the problem solving process; students could be introduced to the power of mathematics in providing an alternative solution or proof to mathematics problems from geometry and calculus. The roots of a complex numbers can be introduced by teaching through problem solving and re-enacting the (simplified) process of how mathematicians discovered complex numbers.

Keywords: problem solving; teaching through problem solving; complex numbers

Introduction

Kissane (2021), in his chapter presented in the 2021 yearbook of the Association of Mathematics Educators (Toh & Choy, 2021) cautioned against excessive attention to the practical usefulness of the subject in an attempt to make mathematics appealing to the wider spectrum of student population. He agreed with the importance of stressing the applicability of mathematics to the real world but advocated the importance of the aesthetic aspect of mathematics for students. This dimension of the "beauty of mathematics" could be interpreted as aiming to develop students' attitude towards mathematics, one of the important dimensions of the Singapore mathematics curriculum.

In the official document of mathematics curriculum by the Singapore Ministry of Education (MOE, for short), problem solving is the heart of the mathematics curriculum from primary to pre-university levels (MOE, 2019). The framework is succinctly summarized pictorially as a pentagon, in which the central position of the pentagon represents problem solving. The pentagon is flanked by five inter-related components: concepts, skills, processes, metacognition and attitudes. To address the affective needs of students (Attitudes) is one important goal of mathematics education, in addition to the other components represented by the other four sides of the Pentagon (Concepts, Skills, Processes and Metacognition).

The aesthetics of mathematics is well documented in the book *Proof without words: Exercises in visual thinking* (Nelson, 1993) published by the Mathematical Association of America. However, mathematicians' appreciation of mathematics likely goes beyond the aesthetic appeal; they are likely enchanted with the beauty of mathematics found in its abstractness and the power of mathematics. Indeed, the abstractness of mathematics can be a powerful tool that allows the theory to develop without the requirement of any particular context and, yet, be applicable to a variety of real world and hypothetical situations. It is indeed this abstraction, which is much appreciated by mathematicians, that is the cause of much difficulty in teaching and learning mathematics (Weissglass, 1990).

In this paper, a proposal to engage students in problem solving with a particular emphasis on problem solving processes in order to provide them with the learning experience within their zones of proximal development (e.g., Fani & Ghaemi, 2011), to develop their abstract mathematical reasoning is presented. The content of our discussion is on complex numbers, a topic that is found in the pre-university syllabus document (MOE, 2017). This paper does not provide an attempt to "concretize" complex numbers or to show the existence of the physical reality of complex numbers (e.g., Antonov, 2013; Vozzo, 2017), but discusses a meaningful engagement of students with complex numbers riding on the spirit of mathematical problem solving.

Mathematical Problem Solving

Leveraging on mathematical problem solving to instill joy in students learning mathematics is not too far-fetched, since problem solving is the main enterprise in the Singapore mathematics curriculum . According to Lester's (1983) definition, a *problem* should not only be one that the solver does not have a readily available solution; he or she should also have the desire to solve it. Thus, instilling joy in problem solving in the students will more readily get them motivated to solve the problems encountered during their career as students.

Classroom anecdotal evidence shows that problem solving has hardly been enacted according to the true sense of the spirit in providing students the opportunity to engage in struggling in solving a non-routine problem. The syllabus documents do not provide sufficient guide on how problem solving could be enacted in the mathematics classrooms, in view of teachers' refrain of time pressure in view of high-stake national examinations. Toh et al. (2008a) presented a model of enactment of problem solving by tracing back to the original spirit of problem solving first proposed by Polya (1954); and how this approach could be used for teaching pre-university mathematics with exemplars from combinatorics (Toh et al., 2008b). A study of how problem solving could be enacted in a secondary school mathematics classroom is presented in Toh et al. (2014). Polya's four-stage problem solving model was represented in the form in Toh et al. (2008a; 2008b) in Fig. 1.

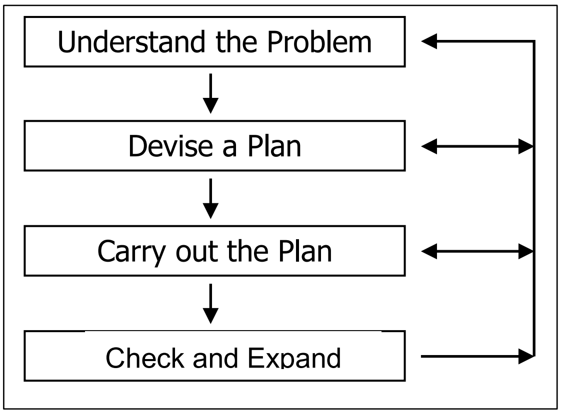

Figure 1. Polya's problem solving model

Polya's fourth stage: check and expand

Although Polya (1954) labelled the fourth stage of his problem solving model as *Look Back,* Toh et al. (2008a; 2008b) modified the name of the fourth stage to *Check and Expand*. A careful reading of Polya's book quickly reveals that the fourth stage includes more than "look back" or checking the correctness or reasonableness of the answers or solutions: it includes looking for alternative solutions; comparing the merits of the various solutions; and also extending a problem. Toh et al. (2011) provides a detailed discussion of expanding a problem with exemplars. Here two aspects of check and expand are discussed next.

Expanding a problem. There are three ways of expanding a problem: adapting, extending and generalizing the problem (Toh et al., 2011). Once the algorithm of solving an original problem is recognized, the solver could *adapt* the problem by altering the conditions of the original problem with the setting of the original problem largely remaining unchanged. Through this process, the solver modifies the algorithm of the solution to the original problem to fit the newly adapted problem, by making slight alternation to the algorithm. *Extending* the problem involves changing the settings of the original problem to produce a new question. In this sense, the solver needs to examine how the original algorithm could be suitably modified to solve the new question with the new settings. *Generalizing* the problem involves creating a new problem in which the original problem can be categorized as a special case of the generalized problem. An illustration of adapting, extending and generalizing a problem (the classical lockers problem) is provided in Toh et al. (2011, p. 11).

Looking for alternative solution to a problem. Looking for alternative solution once an initial solution has been found is part of Polya's fourth stage of check and expand. Educators have begun emphasizing the importance of engaging students to think like a mathematician (e.g., Barlow & Barlow, 2020); looking for alternative (more efficient or elegant) solutions is part of the mathematician's way of thinking (e.g., Stupel & Oxman, 2018). So it becomes undeniable that looking for alternative solution is a crucial part of the check and expand phase of problem solving.

Learning Complex Numbers through Problem Solving

Education literature abounds with studies showing students' difficulty with learning complex numbers. Most of the studies showed that students perceive a dichotomy between the algebraic and geometric representation of complex numbers (e.g., Panaoura et al., 2006). To assist students' learning of complex numbers, educators and researchers have attempted various approaches, such as to facilitate students to visualize the arithmetic of complex numbers (e.g., Soto-Johnson, 2014; Vozzo, 2017); making connections to operation with vectors (e.g., Bostock & Chandler, 1981). Such an "expediency approach" (Driver & Tarran, 1989) has brought students to appreciate the connection of two seemingly unrelated topics, and has obviously served the purpose of equipping students to handle high-stake national examinations well. However, it might not have addressed the problems reflected in common classroom anecdotal accounts that complex numbers (which is understood as the square roots of a negative number) are completely detached from, or at most distantly relate to, the real world (Tan & Toh, 2013).

Introducing complex numbers through extending an algebraic problem

Although students might be comfortable with performing basic arithmetic of complex numbers such as the four basic operations (e.g., Connor et al., 2007), educators could consider providing students with the opportunity to experience the "power" of complex numbers rather than being fixated to the deep-rooted idea that square root of a negative number does not exist, hence the processes of complex numbers belong purely to the realm of imaginary computation. One approach could be to engage the students in developing the stage four part of problem solving by challenging students to extend their knowledge of algebraic manipulation from secondary school to pre-university by expanding a problem.

First, an exemplar is presented on how a problem on secondary school algebraic manipulation could end in the acquisition of complex numbers as a tool through expanding the original problem. Consider the typical question that is used as a problem solving item at the secondary level mathematics classroom on algebraic manipulation presented as problem A below, which is modified from Engel (1998, p. 117):

> **Problem A.** *Prove that the product of two numbers which are the difference of two square numbers is still a number which can be expressed as the difference of two square numbers.*

The solution of this problem is elementary, under which a high school student will likely be able to solve with the repeated application of algebraic expansion and the basic algebraic identity involving the difference of two squares $a^2 - b^2 = (a-b)(a+b)$. The solution goes as follows:

$$\begin{aligned}(a^2-b^2)(c^2-d^2) &= (a-b)(a+b)(c-d)(c+d)\\ &= (a-b)(c+d)(a+b)(c-d)\\ &= \big((ac-bd)+(ad-bc)\big)\big((ac-bd)-(ad-bc)\big)\\ &= (ac-bd)^2-(ad-bc)^2 .\end{aligned}$$

Note that the crux of the solution of problem A does not lie in verifying the identity $(a^2 - b^2)(c^2 - d^2) = (ac - bd)^2 - (ad - bc)^2$, but in *deriving* the form of the right hand side of the equation being equivalent to the left.

Application of Polya's fourth stage. In engaging in the fourth stage of checking and expanding problem A, it was clear that the algorithm of the solution of the above problem relies on the formula of the difference of two squares. Since using the identity on the difference of two squares one has $a^2 - kb^2 = (a - \sqrt{k}b)(a + \sqrt{k}b)$, an adaptation of the problem A goes as follow:

> **Problem B.** *Prove that the product of two numbers of the form $a^2 - kb^2$, where k is a positive number with a and b being integers, is still a number of the form $a^2 - kb^2$, where a and b are integers.*

By going through the solution of problem A, it could be seen that the solution of problem B is parallel to the original solution as follows.

$$\begin{aligned}(a^2 - kb^2)(c^2 - kd^2) &= \left(a - \sqrt{k}b\right)\left(a + \sqrt{k}b\right)\left(c - \sqrt{k}d\right)\left(c + \sqrt{k}d\right)\\ &= (a - \sqrt{k}b)(c + \sqrt{k}d)(a + \sqrt{k}b)(c - \sqrt{k}d)\\ &= \left((ac - kbd) + \sqrt{k}(ad - bc)\right)\left((ac - kbd) - \sqrt{k}(ad - bc)\right)\\ &= (ac - kbd)^2 - k(ad - bc)^2\end{aligned}$$

Indeed, it is obvious that the algorithm to problem A is easily modified to become a solution for problem B (The solution to problem B is in fact a key step in the solution of the Pell's equation, a well-known Diophantine equation.). A natural progression of extending problem B is to consider the case when *k* is negative, and how the above solution to problem B could be extended to negative *k*. Consider the special case when $k = -1$. A natural expansion of problem B would be problem C below.

> **Problem C.** *Prove that the product of two numbers of the form $a^2 + b^2$ is still a number of the form $a^2 + b^2$.*

By attempting to mimic the solution of problem B, and assuming that the "surdic number" $\sqrt{-1}$ exists and which obeys the rule that $\left(\sqrt{-1}\right)^2 = -1$, it appears that the solution of problem B would be able to be replicated, resulting in the following:

$$\begin{aligned}(a^2 + b^2)(c^2 + d^2) &= \left(a - \sqrt{-1}b\right)\left(a + \sqrt{-1}b\right)\left(c - \sqrt{-1}d\right)\left(c + \sqrt{-1}d\right)\\ &= (a - \sqrt{-1}b)(c + \sqrt{-1}d)(a + \sqrt{-1}b)(c - \sqrt{-1}d\\ &= \left((ac + bd) + \sqrt{-1}(ad - bc)\right)\left((ac + bd) - \sqrt{-1}(ad - bc)\right)\\ &= (ac + bd)^2 + (ad - bc)^2\ .\end{aligned}$$

It is instructional for one to verify that $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$ indeed by direct expansion of both sides of the identity. What is amazing in the above solution of problem C was that the number $\sqrt{-1}$ was initially introduced as a "meaningless" entity functionally in order to expand a given problem, but it turns out to solve a problem in the real numbers with the entity $\sqrt{-1}$ exiting at the final solution. Indeed, problem C is the well known rule of complex numbers that the modulus of the product of two complex numbers is the product of the modulus of the two numbers: that is for any two complex numbers z and w, $|zw| = |z||w|$. Such a property of complex numbers could be introduced through a discovery activity based on expanding an algebraic manipulation problem that students first encountered at the secondary level. It could also be introduced as a discovery learning activity in the introductory section on the modulus of complex numbers at the pre-university level.

Appreciating the Power of Complex Numbers through Alternative Solutions

Researchers have agreed that using more than one approach to solve the same problem is essential to development mathematical reasoning (e.g., Polya, 1954; Schoenfeld, 1985; Stupel & Oxman, 2018). Such an approach develops mathematical knowledge and encourages flexibility in one's mathematical thinking (e.g., Silver, 1997; Liekin & Lev, 2007). Further, it is reasonable to believe that to enable students to experience the power of mathematics could also be a desirable result of using more than one approach of solving a problem.

Complex numbers provide a powerful tool that can be used in different fields of mathematics for solving different types of problems. For one, complex numbers are similar to two-dimensional vectors in that (non-zero) complex numbers have both magnitude and direction. However, complex numbers are further endowed with a natural multiplication and division operation which are similar to the operations on surdic expressions. Thus, the use of complex numbers to offer a simpler proof of geometry results is not too surprising. Inviting the students to consider more than one method of doing a geometrical proof, including one method that involves complex numbers, is an important aspect of Check and Expand stage of problem solving in Polya's model.

The challenge for students to think of alternative solution methods could be infused at the check and expand stage of Polya's model, which serves the beginning for students to consider the merits of various alternative methods for the problem which they have managed to solve by this stage. Students could discover the power of complex numbers in offering an alternative (and possibly a simpler) solution to a given problem. Two exemplars are presented for discussion here: one on geometry and another on calculus.

Exemplar 1: Proof in geometry. Consider the well-known Ptolemy's Theorem (e.g., Bradley, 2005; pp. 26, 95; Coxeter & Greitzer, pp. 42, 106), which states that: for any convex quadrilateral $ABCD$, we always have $AC \times BD \leq AB \times DC + DA \times BC$. The equal sign holds true if and only if $ABCD$ is a cyclic quadrilateral.

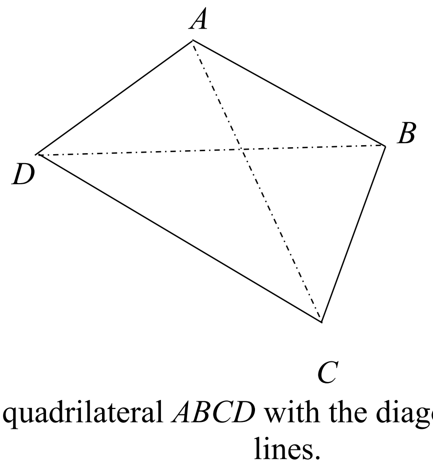

Figure 2. Diagram of a quadrilateral *ABCD* with the diagonals *AC* and *BD* marked in dotted lines.

The classical analytic proof of the Ptolemy's theorem, which makes use of properties of similar triangles and includes additional synthetic lines, is found in most elementary books on plane geometry, hence will not be elaborated here. An alternative proof of Ptolemy's theorem makes use of the algebraic manipulation (the four basic operations) of the complex numbers: if *A, B, C* and *D* represent the complex numbers *a, b, c* and *d* respectively, then by direct computation it is obtained as

$$(c-a)(d-b) = (b-a)(d-c) + (d-a)(c-b).$$

Hence we have

$$\begin{aligned}|(c-a)(d-b)| &= |(b-a)(d-c) + (d-a)(c-b)| \\ &\leq |b-a||d-c| + |d-a||c-b|,\end{aligned}$$

which is equivalent to *AC* x *BD* ≤ *AB* x *DC* + *DA* x *BC* by the triangle inequality, hence the first part of the Ptolemy's theorem is proved. Since $|z_1 + z_2| = |z_1| + |z_2|$ if and only if $z_1 = kz_2$, where $k > 0$, we have the equality in the Ptolemy's theorem holds if and only if $\frac{(b-a)(d-c)}{(d-a)(c-b)} = k$ for some positive number *k*, or equivalently, $\frac{(b-a)(d-c)}{(d-a)(b-c)} = -k$, which is equivalent to the equation in terms of angles $\text{Arg}\left(\frac{b-a}{d-a}\right) + \text{Arg}\left(\frac{d-c}{b-c}\right) = \pi$, that is, the quadrilateral *ABCD* is a cyclic quadrilateral. Indeed, complex numbers offer alternative proofs for many geometry results, which could be found in the exercise section of most chapters of complex numbers in textbooks.

Exemplar 2: An integration problem. Calculus is an important strand in the A-Level mathematics curriculum. Hence integration techniques, including integration-by-parts technique, forms a crucial part of the syllabus. At the pre-university level, students invariably encounter the integration of a function of the product of an exponential and a trigonometric sine or cosine function, of the form $\int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$. The integration of this type of functions involves the application of integration-by-parts twice followed by changing of subjects, or alternatively, forming one pair of simultaneous equations with each equation representing one integration-by-part of the above functions. By converting the trigonometric function into exponential functions involving complex numbers, the integration process can be simplified, as illustrated below,

$$\begin{aligned}\int e^{ax} \cos bx\, dx &= \int e^{ax} \frac{e^{ibx} + e^{-ibx}}{2} dx \\ &= \int \frac{e^{(a+ib)x} + e^{(a-ib)x}}{2} dx \\ &= \frac{e^{(a+ib)x}}{2(a+ib)} + \frac{e^{(a-ib)x}}{2(a-ib)} + C, \qquad (1)\end{aligned}$$

with equation (1) following directly from the standard integration of exponential function $\int e^{kx} dx = \frac{e^{kx}}{k} + C$. The rest of the simplification of the integral expression of (1) uses the basic properties of complex numbers and the Euler's formula. The power of the complex numbers for this case lies in its exponential representation through the Euler's formula, and that it is well known that exponential functions are easier to integrate compared to the product of a trigonometric and an exponential function. Here, the extension of the standard anti-derivative formula for the function e^{kx} to include non-real complex numbers *k* has resulted in a powerful tool.

The above two exemplars could be introduced during or beyond the instructional time with the objective of engaging students to consider alternative solutions when problem solving is infused during the lesson. After properties of modulus and arguments of complex numbers have been introduced, the problems in exemplars 1 and 2 could be introduced as a recap of prior knowledge (secondary school analytic plane geometry for exemplar 1 and integral calculus for exemplar 2). At the Polya's fourth stage of check and expand, the teachers could facilitate the students to relate back to complex numbers, and guide them to appreciate that complex numbers could provide an alternative solution to the problems.

Exploring Complex Numbers from the History of Its Discovery

Researchers have been using the appeal of the personalistic component of the history of mathematics in the teaching of mathematics, and study the impact that this approach would have on the students (e.g., Pavlova et al., 2021). To extend this view, in addition to the personality of the renowned mathematicians, the engagement of students in the mathematical processes (in a simplified form) experienced by mathematicians leading to discovery of new mathematics could have provided an insightful experience for the students. As an illustration, episodes from historical notes show that complex numbers were discovered in mathematicians' attempt of solving cubic equation, rather than quadratic equation. However, the natural sequence of school mathematics curriculum begins with solving quadratic equation leading to cubic equations, which does not explain the discovery of complex numbers, hence making complex numbers appearing "artificial" to students.

Engaging students through the early mathematicians' processes of solving a cubic equation, followed by building on students' problem solving tendency to look back on their solutions, could lead to insight into the discovery of complex numbers. In this section, an illustration is presented on how the notion of complex numbers (as the square root of negative numbers) naturally emerges in the solving of a cubic equation.

Consider the cubic equation $x^3 = 3x + 2$, which is an example of a cubic equation with the term in x^2 suppressed. This is analogous to a quadratic equation in which the term in x is suppressed in the form of $x^2 = a$. This simplified version of a cubic equation was used in Cardan's *Ars Magna*. The solutions of this equation can be found to be x = 2, -1, -1 (the root -1 is a repeated root). Students would have already learnt at their secondary level mathematics how to solve a cubic equation using factor theorem and division algorithms of polynomials (MOE, 2019). On the other hand, the roots of the cubic equation of the form $x^3 = px + q$ can be obtained as

$$\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}, \tag{2}$$

using elementary mathematical tools, such as the sum and product of roots of a quadratic equation (see, for example, Burton, 2003). The mathematical steps to obtain the above solution is within the reach of students at the pre-university level, if one were to construct a hierarchy of learning used by Gagne (1970), as illustrated in Fig. 3.

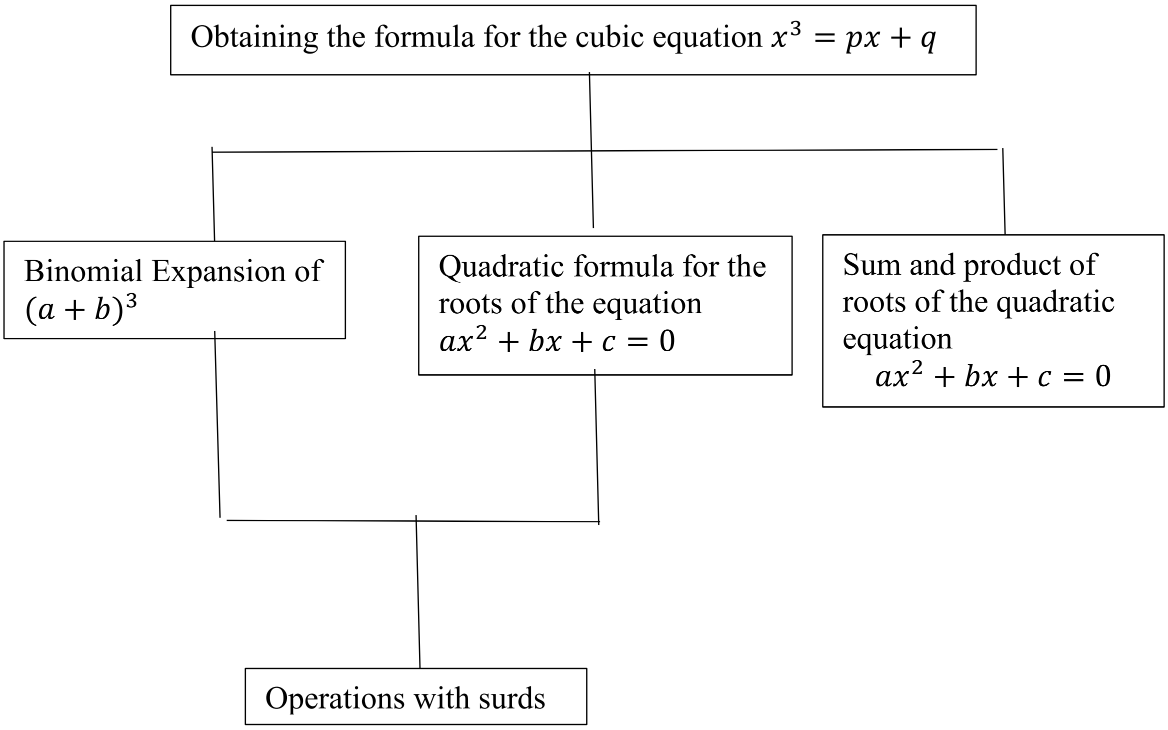

Figure 3. Learning hierarchy for obtaining the formula for the cubic equation with the square term suppressed

Students with the problem solving habit of looking back at the problem (and the solution obtained) will wonder why the expression of the solution itself of (2) appears to have only one solution while the elementary method using factor theorem reveals three solutions. In the particular example of the equation $x^3 = 3x + 2$, the formula yields the solution $x = \sqrt[3]{1} + \sqrt[3]{1}$ (denote this as $a + b$); while the factor theorem yields three solutions 2, -1, -1 for the given equation. This is the first stage of cognitive conflict, which could arise in situations in which students attempt to check their solution using an alternative approach. Teachers could use this as an opportunity to lead their students to develop the desire to resolve the conflict, and reconcile the seemingly different outcomes of the two approaches, and in the process discovers the roots of complex numbers. In particular, the expression $\sqrt[3]{1}$ might refer to different possible numbers in the field of complex numbers.

Consider the historic cubic equation $x^3 = 15x + 4$ that was first explored by Bombelli in 1572. Using the formula (1) for the roots of the cubic equation yields the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

However, one could not dismiss the expression as meaningless, since the use of the factor theorem easily shows that this cubic equation has three real roots 4, $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$. This cognitive dissonance (an expression involving the cube root of a square root could yield three numbers) could easily be used by teachers to tap on their students' exploration of the meaning of the radical notation, which could be different from what they know for the field of real numbers. Such a cognitive dissonance could challenge them to re-consider their prior knowledge (about the field of real numbers) to the new knowledge (of complex numbers):

A. The cube root of a number may represent more than one possible (complex) number?
B. The square root(s) of a negative number may have a meaningful meaning although the square root of a negative number does not exist *as a real number*?

Question A challenges the students' pre-conceived idea that there is only one cube root of a number (e.g. $\sqrt[3]{8} = 2;\ \sqrt[3]{64} = 4$ in the real numbers). Such activities and the questions that could arise in students' mind form a prelude to the roots of complex numbers, and provide a good opportunity to lead them to discover roots of complex numbers.

Enactment of the above lesson for introducing roots of complex numbers

Prior to introducing the notion of roots of complex numbers, the lecturer could (either through face-to-face or flipped lesson) introduce the above activity of discovering roots of a cubic equation as a guided activity with scaffoldings for students to discover the formula (1) as a solution of the equation $x^3 = px + q$. The formula (1) could be represented as $x = a + b$, with the particular relation that $ab = \frac{p}{3}$ (which is a real number, although *a* and *b* may not be real). The students could next be given the first equation $x^3 = 3x + 2$ and asked to solve the equation in their usual way (using factor theorem to obtain all the three real roots). At the check and expand phase, the students could be engaged to use the formula (1) and be challenged to reconcile the apparent contradiction (i.e., three solutions using the factor formula but one solution using the formula).

Providing the next level of hint, the three results $1^3 = 1, \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 = 1; \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = 1$ (or the three roots could be represented in trigonometric form, if they have already learnt de Movire's Theorem) could be provided to scaffold the students to interpret the meaning of $\sqrt[3]{1}$. If the roots are written as $x = a + b$ (the students need to be drawn to the attention that $ab = 1$), so that they could list all the possible pairs of (a, b) and hence the possible values of $a + b$. Get them to compare with the original result (of three solutions of the equation). A similar scaffold can be provided for the students to solve the equation $x^3 = 15x + 4$; it is instructional for the readers to mimic the above process to create the scaffold. This activity easily leads students to appreciate the existence of three square roots of a number in the field of complex numbers.

Teaching through problem solving

In the field of mathematics education research on problem solving, researchers are still using the terms (1) teaching for problem solving; (2) teaching about problem solving; and (3) teaching through problem solving (e.g., Ho & Hedberg, 2005; Stacey, 2005). In particular, teaching through problem solving, the method in which learning of new concepts is the *result* of problem solving processes. Researchers have recognized the positive impact of teaching through problem solving (e.g., King, 2019; Bostic et al., 2016).

The approach of re-enacting the historical episode of discovering complex numbers in solving cubic equation by mathematicians as illustrated in the preceding section is a case of teaching *through* problem solving. Students apply the processes of problem solving which they have acquired in mathematics at the secondary levels of their career as a student to discover new advanced mathematical knowledge of complex numbers. Being cognizant of the usual refrain of school teachers of insufficient curriculum time, teachers could tap on the advantage of flipped learning to gain additional time as much of the elementary activities could be done independently by students without much teacher guidance. This is also a continuation of an emphasis of problem solving at the pre-university level, providing a seamless transition of knowledge acquisition from secondary to pre-university level.

Conclusion

Educators and teachers have felt the real pressure of "not enough time" across different parts of the world. Researchers have also shown that the insufficient time is one main cause of teachers not willing or unable to attempt innovative approaches in teaching (e.g., Manoucherhri, 1999). This is particularly true in the Singapore pre-university institutions which run a lecture-tutorial mode of lesson delivery for most subjects, which are heavy in content. Today, with the introduction of flipped learning, much of the technically involved portions of the activities could be flipped for students' own practice prior to attending the face-to-face lessons. Perhaps mathematical problem solving could ride on the affordance of technology and emerge as a successful flipped learning approach using problem solving.

It could be seen that, in the above discussion, the fundamental principles of much of the above discussions are not beyond the reach of high school students. The objective of the approach outlined above lies in utilizing high school mathematics (or the content which the students are already familiar) and the students' problem solving habits acquired at the primary and secondary levels to learn new content, thereby enabling students to appreciate the content and to experience the power of mathematics.

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