The Mathematician Educator
2022, Vol 3, No. 2, 75-84

Congruence as an Extension to Parity

TAY Eng Guan
National Institute of Education
Nanyang Technological University, Singapore

WONG Willie Han Wah
National Institute of Education
Nanyang Technological University, Singapore

Parity (even and odd) is a feature often utilised in solving a mathematical problem. In Pólya's problem solving model, the solver is encouraged to look back at the solution and pose suitable extensions. Keeping parity as the focus, nice extensions can be posed. In this paper, we give two examples in which seeing parity as a special case of congruence (i.e., as modulo 2) leads to 'nicer' extensions.

Keywords: Pólya, extensions, parity, congruence

Posing new problems in mathematics problem solving

Pólya's (1954) well known 4 phase problem solving model continues to dazzle those who first come to understand it and to be referred to with much respect by those who have been using or teaching it for a number of years. Leong et al. (2011) observed however that the 4th phase of "looking back" is seldom well taught or researched. The 4th phase includes posing new problems by extending, adapting or generalising the original problem. An understanding that is derived from appreciating the 'deep structure' (Schoenfeld, 1985) will often lead to new non-trivial problems that are related to the original problem. In this paper, we show how nice extensions can be obtained for solutions involving parity by seeing parity as a special case of congruence.

Cheung (1992) explained some strategies which utilized heuristics from Pólya's (1954) book How To Solve It for solving Mathematical Olympiad problems. One of these strategies involve noticing the parity (even or odd) of numbers as in the following example and solution from the 1988 China Mathematics Olympiad (Cheung, 1992 p.85).

Problem 1
If $a_1 = 1, a_2 = 2$, and

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n & \text{when } a_n a_{n+1} \text{ is even,} \\ a_{n+1} - a_n & \text{when } a_n a_{n+1} \text{ is odd,} \end{cases}$$

prove that for any natural number *n*, $a_n \neq 0$.

Solution
After calculating the first few values of a_n, we notice a repeating pattern regarding the parity of the terms:

odd, even, odd, odd, even, odd, odd, even, odd, odd, even, …

(Note: It is easy to prove by induction that indeed for $n \in \mathbb{N}, a_{3n} \equiv 1 (\text{mod } 2), a_{3n+1} \equiv 1 (\text{mod } 2), a_{3n+2} \equiv 0 (\text{mod } 2)$.)

Thus, for $n \in \mathbb{N}$,
$a_{3n+2} = a_{3n+1} - a_{3n}$
$a_{3n+2} = 5a_{3n} - 3a_{3n-1} - a_{3n}$
$a_{3n+2} = 4a_{3n} - 3a_{3n-1}$ (*)

Suppose $a_{3m+2} = 0$ for some natural number m. From (*), we have
$a_{3m+2} = 4a_{3m} - 3a_{3m-1}$
$0 = 4a_{3m} - 3a_{3m-1}$
$3a_{3m-1} = 4a_{3m}$.
This implies that a_{3m-1} is a multiple of 4.
Using (*) repeatedly, we have that $a_{3m-1}, a_{3m-4}, a_{3m-7}, \ldots, a_2$ are multiples of 4. Since a_2 is not a multiple of 4, we have a contradiction and thus, for any natural number n, $a_n \neq 0$. ■

Understanding the deep structure of parity will allow one to pose new problems by adapting the original problem. A generalisation is as follows:

Problem 2
If $a_1 = 2^k - 1, a_2 = 2^k$ for some $k \in \mathbb{N}$, p is an odd natural number, and

$$a_{n+2} = \begin{cases} (2^{k+1} + 1)a_{n+1} - pa_n & \text{when } a_n a_{n+1} \text{ is even,} \\ a_{n+1} - a_n & \text{when } a_n a_{n+1} \text{ is odd,} \end{cases}$$

prove that for any natural number n, $a_n \neq 0$.

There are many other interesting problems that can be solved through the perspective of parity. Here are three more examples.

Problem A (Prisoners and Hats Problem)
A warden wishes to give a group of 10 prisoners a chance to be released early. He tells the group that the following day, he will line all 10 of them on a flight of stairs (see Figure 1). Each prisoner faces downwards and can see the heads of the prisoners in front of him. The warden will then put a hat, either black or white, on each of the prisoners' heads, starting from the first prisoner at the top. Thus, each prisoner can see the colour of the hats of the prisoners in front of him but not his own nor those behind him. The prisoners, starting from the first, are then to 'guess' the colour of the hat on their own head, calling out only "Black" or "White"; and they can hear all preceding guesses. If at least 9 of them match what they say with the colour of the hat on their head, the whole group is released early. The warden gives the group the whole night to discuss a strategy. Is there a strategy to guarantee that at least nine of them will get the colour of the hat correct?

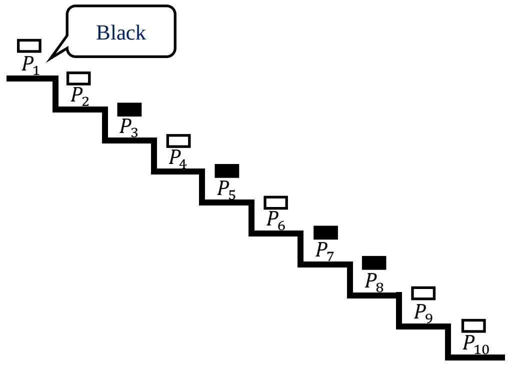

Figure 1. Ten Prisoners wearing black or white hats on a flight of stairs

Problem B (Midpoint Problem)
5 points are chosen randomly on a cartesian plane such that each point has integer coordinates. Prove that there exists a pair such that their midpoint also has integer coordinates.

Problem C (Princess Problem)
There are 5 rooms in a row with a door between every two adjoining rooms. Each room has also a door opening to a corridor. A princess has been imprisoned under a curse by a wicked witch in the row of rooms. She lives in one room each day and moves to an adjoining room the next day at 7 in the morning. A prince wants to rescue the princess and knocks on one door each day at noon. If the princess is in that room, she will open the door and the prince will rescue her. Otherwise, the prince leaves and tries again at some door at the same time the next day. Is there a strategy to guarantee that the prince will rescue the princess if he has 6 days to try?

You may want to try to solve these three problems before you proceed with the rest of the paper.

Congruence as an extension to parity

In the rest of the paper, we will elaborate on our actual attempts to pose new problems based on our understanding of the solution through the perspective of parity. On further reflection, we noticed that parity is a special case of the general modular arithmetic. Extending along this line, we posed new problems which are nice generalisations of Problems A and B. We also explain how an unsatisfactory attempt to generalise Problem C revealed a certain feature of modularity unique to 2.

Problem A *(Solution)*
We first answer the question in the affirmative. Note that the first prisoner has no way of knowing the colour of his hat to guarantee him a correct guess. Thus, a successful strategy would necessitate that his guess conveys some information to the other prisoners so that the latter can deduce the colour of their hats. Such information should be "binary" in nature to be encoded to his guess, "Black" or "White". Therefore, a possible strategy is that the first prisoner calls out "Black" (respectively, "White") if the number of black hats he sees is even (respectively, odd). We represent this in Figure 2.

1st prisoner's call out	**Number of black hats**
Black	even
White	odd

Figure 2. Strategy based on the parity of the number of black hats

Suppose the first prisoner calls out "Black". The second prisoner, being aware of the number of black hats in front of him, will then be able to deduce the colour of his hat and calls out a correct guess. Specifically, if he sees an odd (respectively, even) number of black hats, then he must be wearing a black (respectively, white) hat. Subsequent prisoners can deduce the colours of their hats from the preceding guesses similarly. We leave it to the reader to verify the case in which the first prisoner calls out "White".

Clearly, the actual word called out by the first prisoner is immaterial. He could have called out "W_1" or "W_2" instead of "Black" and "White", provided his cellmates understood the case each word represents.

Problem A *(Extension using parity)*
It is natural to consider the extension of Problem A in which each hat is of one of $n \geq 2$ colours, say colours $1, 2, \dots, n$. To this end, we apply a similar strategy to a less restrictive problem, where the first prisoner is given 2^{n-1}distinct words to call out, say $W_1, W_2, \dots, W_{2^{n-1}}$, instead of n distinct words. The other prisoners would still be guessing the colour of their own hats.

Since the number of hats with colour i, for $i = 1, 2, \dots, n-1$, is either even or odd, there is a total of 2^{n-1} possible cases. Then, the prisoners can agree to an assignment of words to the cases (i.e. bijection), say as shown in Figure 3.

1st prisoner's call out	***colour 1***	***colour 2***	***colour 3***	$\cdots$	***colour*** $\boldsymbol{n-1}$
W_1	even	even	even	$\cdots$	even
W_2	odd	even	even	$\cdots$	even
W_3	even	odd	even	$\cdots$	even
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$W_{2^{n-1}}$	odd	odd	odd	$\cdots$	odd

Figure 3. Strategy based on the parity of the number of hats with colours $1, 2, \dots, n-1$

Suppose the first prisoner calls out "W_1". Then, the second prisoner is informed that the first prisoner saw an even number of hats with colour i, for all $i = 1, 2, \dots, n-1$. If he observes the same too, then his hat must be of colour n. Otherwise, he observes that for exactly one $j \in \{1, 2, \dots, n-1\}$, the number of hats with colour j is odd, in which case, his hat must be of colour j. A similar argument follows for the remaining prisoners. We leave it to the reader to verify the remaining cases in which the first prisoner calls out "W_i", $i = 2, 3, \dots, 2^{n-1}$.

In the above, we managed to generalize 'naturally' the original strategy, based on parity, by giving the first prisoner a privilege of 2^{n-1} distinct words to call out.

***Problem A** (Extension using general congruence)*
For the extension with n colours, we show now that it is possible to construct an effective strategy even if we restrict the first prisoner to a choice of n words (or simply the n colours). For $i = 1, 2, \dots, n$, assign colour i to the i^{th} congruence class modulo n, i.e. $f(i) \equiv i \ (mod\ n)$. For $j = 1, 2, \dots, 10$, let the colour of the j^{th} prisoner's hat be c_j. Note that the k^{th} prisoner can determine the sum $\sum_{j=k+1}^{10} f(c_j)$ based on what he sees. Suppose further $\sum_{j=k}^{10} f(c_j) \equiv a_k \ (mod\ n)$ for $k = 2, 3, \dots, 10$.

The first prisoner computes $\sum_{j=2}^{10} f(c_j) \equiv a_2 \ (mod\ n)$ and calls out "colour a_2" The second prisoner, being able to determine $\sum_{j=3}^{10} f(c_j) \equiv a_3 \ (mod\ n)$, can deduce the difference $f(c_2) \equiv (a_3 - a_2) \ (mod\ n)$, and thus his hat's colour. In general, the k^{th} prisoner can determine the sum $\sum_{j=k+1}^{10} f(c_j)$ based on what he *sees* on the heads of the prisoners in front of him, the sum $\sum_{j=2}^{10} f(c_j)$ based on what he *hears* from the first prisoner, and the sum $\sum_{j=2}^{k-1} f(c_j)$ based on what he *hears* from the other prisoners behind him. Thus, he can calculate his hat colour $f(c_k) = \sum_{j=2}^{10} f(c_j) - \sum_{j=2}^{k-1} f(c_j) - \sum_{j=k+1}^{10} f(c_j)$ (mod n).

***Problem B** (Solution)*
The midpoint of two points (x_1, y_1) and (x_2, y_2) is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. After trying with some examples, one would observe that the average $\frac{x_1+x_2}{2}$ of two numbers x_1 and x_2 is an integer if and only if both numbers are of the same parity, i.e. both are even or both are odd. Hence, partition all points into the four types, (odd, odd), (even, odd), (odd, even) and (even, even). Given any 5 distinct points, there exist $\left\lceil \frac{5}{4} \right\rceil = 2$ points belonging to the same type by the Pigeonhole Principle; the midpoint of this pair has integer coordinates.

***Problem B** (Extension using parity)*
We may extend the problem and solution from the cartesian plane $\mathbb{R}^2$ to $\mathbb{R}^n$. Given any $2^n + 1$ distinct points with integer coordinates in Cartesian n-space, prove that there exist two of them whose midpoint has integer coordinates. Similar as before, the midpoint of two distinct points $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ is $(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \dots, \frac{a_n+b_n}{2})$.

We employ the same strategy of considering the parity of each coordinate. Since each of the n coordinates, a_i, is either even or odd, there is a total of 2^n types of points. Hence, given any $2^n + 1$ distinct points, there exist $\left\lceil \frac{2^n+1}{2^n} \right\rceil = 2$ points belonging to the same type by the Pigeonhole Principle. It follows that the midpoint of this pair has integer coordinates.

***Problem B** (Extension using general congruence)*
In what follows, we generalise the notion of midpoint to an "m^{th}section point" in the cartesian plane $\mathbb{R}^2$ using congruence classes. For $m \geq 2$, let the m^{th}section point of two distinct points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ be $(\frac{x_2+(m-1)x_1}{m}, \frac{y_2+(m-1)y_1}{m})$ (see Figure 4 for an example where

$m = 3$). Given any $m^2 + 1$ distinct points with integer coordinates in the Cartesian plane, prove that there exist two of them whose m^{th}section point has integer coordinates.

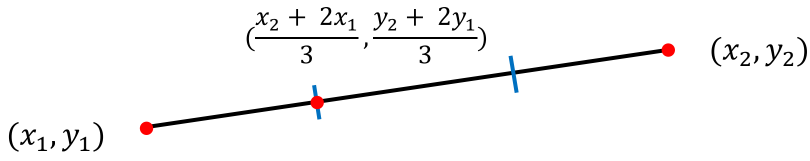

Figure 4. 3rd section point of two points.

We extend the above strategy by considering the congruence class modulo m of each coordinate. Since there are m congruence classes for each of the two coordinates, it follows that there is a total of m^2 types of points. Hence, given any $m^2 + 1$ distinct points, there exist $\left\lceil \frac{m^2+1}{m^2} \right\rceil = 2$ points, say (x_1, y_1) and (x_2, y_2), belonging to the same type, say $(r_1 \ (mod\ m), r_2 \ (mod\ m))$ for some $r_1, r_2 = 0, 1, \ldots, m-1$, by the Pigeonhole Principle. Since $x_2 + (m-1)x_1 = m \cdot r_1 (mod\ m) = 0\ (mod\ m)$ and $y_2 + (m-1)y_1 = m \cdot r_2 (mod\ m) = 0\ (mod\ m)$, the m^{th}section point of these 2 points has integer coordinates.

Actually, the statement is true if we replace " $(\frac{x_2+(m-1)x_1}{m}, \frac{y_2+(m-1)y_1}{m})$ " by "$(\frac{kx_1+(m-k)x_2}{m}, \frac{ky_1+(m-k)y_2}{m})$ for any $k = 1, 2, \ldots, m-1$". We leave the similar proof to the reader.

Problem C *(Solution)*
Let us work for n rooms, with $n \geq 3$. Let the rooms be $R_{j'}$, where $j' = 1, 2, \ldots, n$. Then if the prince knocks on $R_{j'}$ on Day $j' - 1$ and Day $2n - j' - 2$, for $j' = 2, 3, \ldots, n$–1, the princess will be met on some day within $2(n-2)$ days.

We first show a simple graphic proof that the strategy works.

Represent the situation as a $2(n-2)$ by n chessboard (see Figure 5 when $n = 5$), with the rows representing the days and the columns the rooms. Colour the cells black or white as in a chessboard. Observe that on each day the princess' room movement (say, she is in R_j on Day k,) has the same colour (i.e. same parity for $j + k$). So the prince must select rooms which intersect all possible monochromatic paths, such as the strategy shown by the X's, which can be seen as representing a black diagonal wall blocking all monochromatic black paths (bold lines) followed immediately by a white diagonal wall blocking all monochromatic white paths (dotted lines). The observation also invites seeing other (albeit similar) effective strategies.

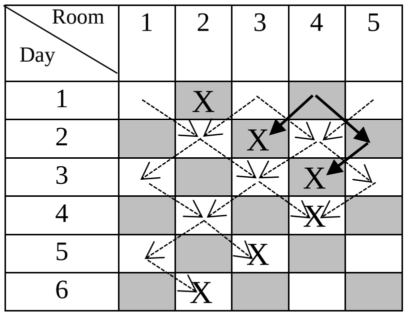

Figure 4. Graphic proof of strategy for Problem C when n = 5

For a more rigorous proof, suppose the princess is initially in room R_i, where $i = 1, 2, \ldots, n$. We will show that the prince's strategy will ensure that the princess will be encountered. As mentioned in the previous paragraph, if the princess is in R_j on Day k, then $j + k$ is either even or odd for all days. With the assumption that the princess is in R_i on Day 1, we show that $i + j + k$ is odd for all days; this will also give a neater argument of the prince's strategy.

Claim 1: If the princess is in R_j on Day k, then $i + j + k$ is odd.
Proof On Day 1, the princess is in R_i, so $i + j + k = i + i + 1 = 2i + 1$, which is odd. This is used as the base case for an inductive proof. For each day that passes, k increases by 1, and j is increased or decreased by 1, so $i + j + k$ either remains constant or increases by 2 and thus remains odd.

Claim 2: If the prince knocks on $R_{j'}$ on Day k, then $j' + k$ is odd for $k \leq n - 2$ and even otherwise.
Proof If $k \leq n - 2$, then $k = j' - 1$. Thus, $j' + k = j' + j' - 1 = 2j' - 1$, which is odd. If $k > n - 2$, then $k = 2n - j' - 2$. Thus, $j' + k = j' + 2n - j' - 2 = 2n - 2$, which is even.

Case 1: i is even, i.e. the princess starts in an even-numbered room.
We show that the prince encounters the princess in one of the first $n - 2$ days. By Claim 1, if the princess is in R_j on Day k, then $j + k$ is odd. By Claim 2, if the prince knocks on $R_{j'}$ on Day k, then $j' + k$ is odd for $k \leq n - 2$. Thus, j and j' have the same parity. If $k = 1$, then $j + k$ is odd implies that $j \geq 2 = j'$. If $j = j'$, the princess is encountered on Day 1. Otherwise, to maintain parity, $j \geq j' + 2$ when $k = 1$. For each day that passes in the first $n - 2$ days, k increases by 1, j is increased or decreased by 1, and j' is increased by 1, so $j = j'$ or $n \geq j \geq j' + 2$. Since n is finite, and $j' = n - 1$ when $k = n - 2$, it follows that $j = j'$ for some $k \leq n - 2$ and the princess is encountered.

Case 2 : i is odd, i.e. the princess starts in an odd-numbered room.
We show that the prince encounters the princess in one of the last $n - 2$ days. By Claim 1, if the princess is in R_j on Day k, then $j + k$ is even. By Claim 2, if the prince knocks on $R_{j'}$ on Day k, then $j' + k$ is even for $n - 1 \leq k \leq 2(n - 2)$. Thus, j and j' have the same parity. If $k = n - 1$, then $j + k$ is even implies that $j \leq n - 1 = j'$. If $j = j'$, the princess is encountered on Day $(n - 1)$. Otherwise, to maintain parity, $j \leq j' - 2$ when $k = n - 1$. For each day that

passes in the last $n-2$ days, k increases by 1, j is increased or decreased by 1, and j' is decreased by 1, so $j=j'$ or $1 \le j \le j'-2$. Since n is finite, and $j'=2$ when $k=2(n-2)$, it follows that $j=j'$ for some $n-1 \le k \le 2(n-2)$ and the princess is encountered.

Problem C *(Extension using parity)*
Now, we consider the extension in which the princess moves to the 2nd adjacent room each day, i.e. if she is currently in R_j, for some $j=3,4,\dots,n-2$, then she either moves to R_{j-2} or R_{j+2}; if she is in R_j, $j=1,2$, (respectively, $j=n-1,n$), then she must move to R_{j+2} (respectively, R_{j-2}), i.e., "bouncing" from the end rooms is not allowed. We show that there is a strategy for the prince to rescue the princess in $2(n-4)$ days by considering the congruence classes modulo 4. For simplicity, we assume n is a multiple of 4; otherwise, the strategy can be modified accordingly. The prince knocks on R_i on Days $\frac{i-1}{2}$ and $n-\frac{i+5}{2}$, for $i=3,5,\dots,n-3$, and R_j on Days $n-5+\frac{j}{2}$ and $2n-6-\frac{j}{2}$, for $j=4,6,\dots,n-2$.

Note that there are two cases here; the princess in R_j either alternates daily between $j \equiv 1\ (mod\ 4)$ and $j \equiv 3\ (mod\ 4)$, or between $j \equiv 0\ (mod\ 4)$ and $j \equiv 2\ (mod\ 4)$. The essence of this strategy is to apply the notion of "parity" in each case. In the former case, we may perceive the rooms R_j with $j \equiv 1\ (mod\ 4)$ as the "odd" rooms and $j \equiv 3\ (mod\ 4)$ as the "even" rooms. Furthermore, the rooms R_j with $j \equiv 0\ (mod\ 4)$ or $j \equiv 2\ (mod\ 4)$ can be ignored since the princess does not enter any of them in this case. After which, the "black and white diagonal walls" ensure, as before, that the prince finds the princess. Similarly, in the second case, the rooms R_j with $j \equiv 0\ (mod\ 4)$ and $j \equiv 2\ (mod\ 4)$ corresponds to the "odd" and "even" rooms. We leave the verification to the reader and illustrate in Figure 6 the graphic proof when there are $n=8$ rooms.

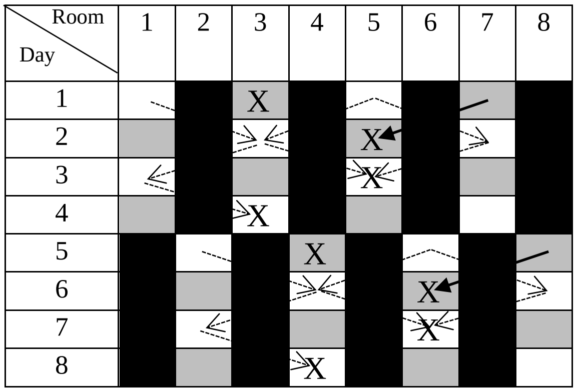

Figure 5. Graphic proof of strategy when $n=8$

Next, suppose the princess moves to the m^{th} adjacent room each day, $m \ge 2$. It is easy to emulate the above strategy by considering the congruence classes modulo $2m$. There will be m cases. Each case corresponds to the prince tracing a black and a white diagonal walls in R_j, $j \equiv r\ (mod\ 2m)$ or $j \equiv r+m\ (mod\ 2m)$ for some $r=0,1,\dots,m-1$. In essence, we have adapted the original strategy to apply the idea of "parity" in each case. Let us return to the original solution (for $m=1$) to explain why parity works. If the princess is in R_j on Day k, then denote the sum of the day number and room number as

$$T(j,k) = j + k \qquad (1)$$

As each day passes, k increases by 1, while j either increases or decreases by 1. In both cases, the value of $T(j,k)$ remains invariant congruent modulo 2 everyday, i.e.

$$T(j+1,k+1) \equiv T(j,k) \equiv T(j+1,k-1)\ (mod\ 2). \qquad (2)$$

Observe in Figure 5 that the black and white diagonal wall corresponds to the cases $T(j,k) \equiv 1\ (mod\ 2)$ or $T(j,k) \equiv 0\ (mod\ 2)$. That is, the black (respectively, white) diagonal wall ensures the prince rescues the princess with $T(j,k) \equiv 1\ (mod\ 2)$ (respectively, $T(j,k) \equiv 0\ (mod\ 2)$).

Problem C *(Extension using general congruence)*
Consider the extension in which the princess moves to the m^{th} adjacent room each day, $m \geq 2$. Suppose we try to extend the original solution, which used congruence classes modulo 2 when $m = 1$, using congruence classes modulo $m + 1$. Day by day (and as per (1)), the day number k increases by 1, while the room number j either increases or decreases by m. However, the property (2) does not hold for $m \geq 2$, i.e.

$$T(j+1,k+m) \equiv T(j,k) \not\equiv T(j+1,k-)\ (mod\ m+1).$$

Since $T(j,k)$ may belong to a different congruence class modulo $m + 1$ each day, a similar strategy of using a black and a white diagonal walls to cover the possibility $T(j,k) \equiv r\ (mod\ m)$ for some $r = 0, 1, \ldots, m$, would not work. Though this deliberation of congruence classes modulo m is not successful, it elucidates the key difference in the underlying mechanisms between $m = 1$ and $m \geq 2$, namely Property (2).

We note this unique feature of modulo 2, that is, $x - (rk+1) \equiv x + (rk+1)$ (mod r) if and only if $r = 2$. An extension of the original strategy will solve the problem for the princess moving odd number of steps (i.e. to the $(2k+1)^{\text{th}}$ adjacent room) each day if "bouncing" from the end rooms is not allowed.

Pedagogical implications

This article examines the affordance of seeing parity as a special case of congruence in relation to extensions to problems. It gives two examples where the extensions are much more instructive compared to those which were limited to a parity perspective. It also gives an example where indeed the special characteristic of modulo 2 makes a problem unique and not extendible.

Big Ideas in school mathematics are overarching concepts that occur in various mathematical topics in a syllabus (Tay, 2019). Knowing these will guide teachers to help students develop a better understanding of mathematics, by making visible the central ideas, the coherence and connection across topics and the continuity across levels. Parity is a Big Idea that links the counting numbers in primary school to the topics of factorization and divisibility in secondary school. Congruence is a Bigger Idea in which parity is a part. The progression from 1 to 2 can be quite startling, such as from lines to quadratic curves. To stop at quadratics without pushing a little into cubics and higher powers is a little like stopping at parity and not going into general congruences. For example, one could extend the number of roots of linear polynomials (1) to quadratic (0, 1, 2) to cubic (1, 2, 3) and onward. One can also wonder about the intersections of 2 distinct lines (0, 1) to 2 quadratic curves (0, 1, 2) to cubic 2 cubic curves (0, 1, 2, 3) and onward. Just as our examples show, seeing 3 and more as an extension of 2 is quite easy and enlightening.

Acknowledgements

We would like to thank the reviewers for their kind and useful comments and suggestions.

References

Cheung, P. H. (1992). Problem solving strategies: Research findings from Mathematics Olympiads. *Mathematical Medley*, 20(2), 79-98.

Leong, Y.H., Tay, E.G., Toh, T.L., Quek, K.S., & Dindyal, J. (2011). Reviving Pólya's "Look Back" in a Singapore school. *Journal of Mathematical Behavior*, 30(3), 181-193.

Pólya, G. (1954). *How to solve it.* Princeton: Princeton University Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Tay, E. G. (2019). Teaching Mathematics: A modest call to consider Big Ideas. In Toh, T. L. & Yeo, J. B. W. (Eds.), *Big ideas in mathematics Yearbook 2019* (pp. 129-138). Singapore: World Scientific.

Author
TAY, Eng Guan (engguan.tay@nie.edu.sg),
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616

WONG, Willie Han Wah (williewong088@gmail.com),
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616