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Collaborative Creativity in Proving: Adapting a Reflection Tool for Group Use

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Creativity is central to mathematics and mathematics education. One hindrance to research in mathematical creativity is the complex nature of defining and measuring creativity. Some efforts have been made to study mathematical creativity at the K-12 level, but just recently researchers have begun exploring mathematical creativity at the undergraduate level. Proof is essential to an undergraduate mathematics education, and as university mathematics classrooms evolve to incorporate more active and collaborative learning, it is imperative to understand the relationship between collaboration and creativity in proving. This study seeks to apply the Creativity-In-Progress Reflection (CPR) on Proving (Savic et al., 2017) tool in a collaborative setting and evaluate its effectiveness in presenting a holistic image of a group's creative proving process. Findings of this evaluation led to the suggestion of three modifications to the CPR on Proving for use in a collaborative setting.

Keywords: Mathematical creativity, collaboration, reflection

Introduction and Background

Creativity is a central tenet of mathematics, and mathematical creativity is a rapidly growing research area. Since the year 2000, there are over 7800 items on Google Scholar that contain the phrase "mathematical creativity," and about 2200 of these were published in the last two years alone. A Google Scholar search restricted to before the year 2000 reveals that there are only 793 such results. Mathematical creativity has been recognized as an important part of mathematics education. The CUPM Curriculum Guide to Majors in the Mathematical Sciences (Schumacher & Seigel, 2015) asserted encouraging students to think creatively is essential to successful mathematics major courses. Neglecting mathematical creativity in mathematics curricula denies students from truly understanding what mathematics is and from seeing the beauty of mathematics (Mann, 2006).

Defining mathematical creativity has been a difficult task for scholars in mathematics and mathematics education (Karakok et al., 2015; Mann, 2005, 2006; Nadjafikhah et al., 2012; Sriraman, 2004). Definitions of creativity vary depending on whether creativity is characterized as domain-general or domain-specific, a process or an end product, and relative or absolute (Savic et al., 2017). Definitions of mathematical creativity have been formulated by adapting domain-general definitions from psychology (e.g., Savic, 2016; Sriraman, 2005, 2017), analyzing mathematician's beliefs and creative processes (e.g., Karakok et al., 2015; Sriraman, 2004), and comparing the types of reasoning used in undergraduate mathematics courses (e.g., Bergvist, 2007; Lithner, 2008; Mac an Bhaird et al., 2016). Lack of a central definition of mathematical creativity has created a challenge in measuring creativity and thus pursuing empirical research in mathematical creativity.

Guilford's (1959) characterization of creativity aimed to present testable factors of creativity and has been commonly used as a framework for measuring and defining creativity. The four aspects of creativity, as suggested by Guilford, are fluency, flexibility, originality, and elaboration. This characterization inspired the Torrance Tests of Creative Thinking (TTCT; Torrance, 1966, 1974) to measure domain-general creativity, the similar Mathematical Creativity Test (MCT; Kattou et al., 2013) to measure domain-specific mathematical creativity, and many others (e.g., Leikin & Elgraby, in press; Levav-Waynberg & Leikin, 2012; Siswono, 2010).

Despite agreement that proof is an essential aspect of learning mathematics at the undergraduate level (Karakok et al., 2015; Sriraman, 2004), tools to measure and evaluate creativity in proof writing have been sparsely explored. Only recently has there been an effort to formally assess students' creativity in proving (Savic et al., 2014). The Creativity-In-Progress Rubric (CPR) on Proving (Figure 1; Savic et al, 2017), originally named the Creativity in Proof Rubric (Savic et al., 2014), was inspired by a domain-general interdisciplinary rubric on creativity (Rhodes, 2010, as cited in Savic et al., 2014). Savic et al. developed the CPR on proving by combining ideas from this rubric with information gathered from interviews with research mathematicians (Karakok et al., 2015), Guilford's (1959) fluency, flexibility, and originality, and aspects of the Torrance Tests of Creativity (Torrance, 1966).

The CPR on proving assumes creativity to be domain-specific, a process, and relative. The most recent published version of the CPR on proving (Savic et al., 2017) presents two categories: making connections and taking risks, which are split into subcategories. Each subcategory is assessed on three general levels: beginning, developing, or advancing, but these levels provide structure for a continuum rather than strict categories. The level assigned characterizes the proving process during a single proof of an individual student, not the final product of the proof attempt. The CPR on proving has been used as a formative assessment tool (e.g., El Turkey et al., 2018) and student reflection tool (e.g., Omar et al., 2019; Tang et al., 2017). In order to align the CPR with these goals rather than summative assessment, the creators of the CPR, the Creativity Research Group (Creativity Research Group, 2022), have since changed the title from Creativity-in-Progress Rubric to Creativity-in-Progress Reflection in recent work. The Creativity Research Group has also suggested future research on the influence of socialization and collaboration on creativity in proof (Savic, 2016) using the Creativity-In-Progress Rubric.

MAKING CONNECTIONS:	Beginning	Developing	Advancing
Between Definitions/Theorems	Recognizes some relevant definitions/theorems from the course or textbook with no attempts to connect them in their proving	Recognizes some relevant definitions/theorems from the course and attempts to connect them in their proving	Implements relevant definitions/theorems from the course and/or other resources outside the course in their proving
Between Representations[1]	Provides a representation with no attempts to connect it to another representation	Provides multiple representations and recognizes connections between representations	Provides multiple representations and uses connections between different representations
Between Examples	Generates one or two specific examples with no attempt to connect them	Generates one or two specific examples and recognizes a connection between them	Generates several specific examples and uses the key idea synthesized from their generation
TAKING RISKS:	**Beginning**	**Developing**	**Advancing**
Tools and Tricks	Uses a tool or trick that is algorithmic or conventional for the course or the student	Uses a tool or trick that is model-based or partly unconventional for the course or the student	Creates a tool or trick that is unconventional for the course or the student
Flexibility	Attempts one proof technique	Acknowledges the possibility of different proving approaches, but attempts no further examination	Acts on different proving approaches
Perseverance	Begins to engage with proving	Continues to engage with surface level features but not with the key ideas	Continues to engage with the key ideas
Posing Questions	Recognizes a question should be asked, but does not formulate a question	Poses questions clarifying a statement of a definition or theorem	Poses questions about reasoning within a proof
Evaluation of the Proof Attempt	Checks work locally	Recognizes a successful or unsuccessful proving attempt	Recognizes the key idea that makes the proving attempt successful or unsuccessful

[1] We define a *mathematical representation* similar to NCTM's (2000) definition. It includes written work in the form of diagrams, graphical displays, and symbolic expressions. We also include linguistic expressions as a form of lexical or oral representation. For example, a student can use the lexical or oral representation, "the intersection of sets A and B"; a Venn Diagram to depict his/her mathematical thinking; a symbolic representation $A \cap B$; or set notation $\{x | x \in A \text{ and } x \in B\}$ (which is also a symbolic representation). Note the last two representations are in the same category, e.g. symbolic, but they are still considered two different representations.

Figure 1. The Creativity-In-Progress Rubric on Proving
From "Formative Assessment of Creativity in Undergraduate Mathematics: Using a Creativity-in Progress Rubric (CPR) on Proving" by Savic, M., Karakok, G., Tang, G., El Turkey, H., and Naccarato, E., 2017, In R. Leikin & B. Sriraman (Eds.), Creativity and giftedness (pp. 23–46). https://doi.org/10.1007/978-3-319-38840-3.

Group Creativity

Students in a classroom do not necessarily act on their own. Students' ideas and creative contributions are influenced through collaboration with peers and instructors. Many mathematics classrooms employ group work as a means to improve learning; Slavin (1996) even called collaborative learning, "one of the greatest success stories in the history of educational research" (p. 43). In my review of undergraduate mathematical creativity research, however, studies have almost exclusively investigated the creativity of an individual rather than the creativity of a collaborative group.

Group creativity is the generation of creative ideas by groups when the interactions and inputs of several people are considered (Levenson, 2011). Investigations on group creativity have indicated that a creative product may not necessarily be improved by collaboration (Paulus et al., 2000). Paulus and Yang (2000) claimed that poor outcomes of group creativity may be a result of inattentiveness of group members, difficulty generating ideas while listening to the ideas of others, and lack of incubation time during collaborative work. In collaborative settings wherein diverse individuals work toward a common goal, different backgrounds and knowledge from a diverse group may contribute to the creative product, but it is also possible for diversity to be too wide and deter agreement on a common solution (Kurtzberg & Amabile, 2001).

In order to describe interactions among students and how they collaboratively reach, or fail to reach, a common goal in proving a mathematical statement, I use a framework of *participatory intersubjectivity*. A traditional definition of intersubjectivity would describe intersubjectivity as an overlapping set of individual subjectivities or when more than one individual shares the same subjective perspective of reality (Matusov, 1996). Matusov (1996) introduced an alternative definition of intersubjectivity, called participatory intersubjectivity, which defined intersubjectivity to be the "process of coordination of individual contributions to joint activity rather than as a state of agreement" (Matusov, 1996, p. 34). A lens of participatory intersubjectivity enables one to account for how something new is created through group collaboration (Sawyer, 2019).

Sawyer (2019) used intersubjectivity as a framework to analyze and describe improvisational theatre performances, and the framework allowed him to see that "Although each actor may have a rather different interpretation of what is going on and where the scene might be going, they can nonetheless proceed to collectively create a coherent dramatic frame" (Sawyer, 2019, p. 578). A group of students engaging in collaborative mathematical proving is quite similar to a troupe of improvisational actors. The students are given a prompt in the form of a given task or statement to prove, and each student may have a different interpretation of what should happen within the proof or a different vision for the end product of the proof, but these perspectives and ideas evolve as they act and react to their peers during the collaborative process. Eventually, students create a coherent proof despite the differences among their subjective experiences.

Goals of This Study

This study aims to investigate the characteristics of collaborative creativity in proving in order to adapt a creativity reflection tool for group use. I seek to answer the question: how can creativity in proving be characterized in a collaborative setting? I wish to examine the ability of the existing individual CPR on Proving (Savic et al., 2017) to depict a holistic representation of the creative process of a group. This study's definition of mathematical creativity aligns with that of Savic et al. (2017); I assume that mathematical creativity is domain specific, relative in terms of context and background, and assessed as a process rather than a product.

Methods

Data for this project was collected from an undergraduate introductory course in proving. This data was collected as a part of a larger project on authority in collaborative proof writing. The data used in this study consisted of two 25-minute video recordings of group work during a class meeting in the first week of the course. Each video captured a group of three students working together to prove a statement provided by the instructor. This course was facilitated in a collaborative learning environment, and the instructor emphasized active learning and group work. The decision to investigate this course and section was a convenience sample, but the two videos investigated in this study were selected using a matched-comparison sampling technique. These specific videos provided unique insight to my question as (1) this group work occurred early in the course, so students were unfamiliar with proving and most contributions in terms of proving strategy could be assumed to be novel, and (2) both groups are asked to prove the same task and thus their proving processes can be directly compared.

Students in the class were asked to work in assigned groups of three to prove the statement: the product of consecutive twin primes is one less than a perfect square. The instructional goal for this lesson was to encourage students to start thinking like mathematicians and see the power of generalization as a tool for proving. Although this statement was relatively simple to prove, previous research (Smith, 2006) has demonstrated this task can elicit many different strategies in proof construction. When the class was presented with the proving task, the instructor encouraged students to consult previously learned definitions (prime number, twin primes, perfect square, etc.) and to use alternative representations such as diagrams, pictures, language, and symbols. Students worked in their groups of three to prove the given statement and occasionally asked questions of the instructor or graduate assistants.

The two video recordings were first transcribed and then coded according to a protocol coding method (Saldana, 2013) using the Creativity-In-Progress Rubric (CPR) on Proving (Savic et al., 2017). The codes were predetermined according to eight subcategories of the CPR on Proving. The CPR is divided into two categories: *Making Connections* and *Taking Risks*. The subcategories of the Making Connections category produced three corresponding codes: (1) definitions/theorems, (2) representations, and (3) examples. The subcategories of the Taking Risks category yielded the remaining five codes: (4) tools and tricks, (5) flexibility, (6) perseverance, (7) posing questions, and (8) evaluation of the proof attempt.

For codes 1-4, the video data was coded for each instance of the group employing the use of a definition/theorem, representation (graphical, algebraic, symbolic, etc.), example, and tool or trick, respectively. Similarly, for code 7, posing questions, each video was coded for each instance of a participant posing a question. Code 5, flexibility, refers to the ability to change direction during the proving process or attempt the proof using a variety of strategies. To code flexibility, instances were coded wherein a new proving strategy was proposed or pursued. Instances when groups or group members showed great persistence or withdrew from the proving process were coded as code 6, perseverance. Finally, moments in which the group evaluated their progress, ideas, or a contribution of a group member were coded as code 8, evaluation of the proof attempt.

The goal of this coding strategy was to identify moments in which the group engaged in creative proving as defined by the CPR. This allowed the group to be viewed as a cohesive unit of analysis and assess their proving process in each subcategory on the spectrum of *beginning, developing,* and *advancing* (see Figure 1 for descriptions of the continuum levels in each subcategory). After coding each group's proving process, I holistically analyzed the instances in each code and placed each group on the CPR continuum in each subcategory, aligning with the levels and examples of individual implementation of the CPR as in Savic et al. (2017).

The final stage of analysis aimed to assess the effectiveness of the CPR on Proving in telling the story of each group's creative proving process. In this stage, I rewatched the video data and observed important moments and themes which influenced the group's process and were not conveyed by the CPR on Proving for the group.

Findings

Video recordings of two groups working together on the same proving task were analyzed. Both groups were asked to prove the statement: the product of consecutive twin primes is one less than a perfect square. A notable feature of this statement is that a more general statement is true: the product of any two numbers with a difference of two is one less than a perfect square. Group 1 consisted of three students (all names are pseudonyms): Nathan, Danvir, and Matthew; Group 2 also consisted of three students: Susan, Mason, and Craig. In this section, I present the assessment of each group's work according to the Creativity-In-Progress Rubric on Proving for each subcategory and provide examples in support of their evaluations. I then describe the findings from my second phase analysis of the effectiveness of the CPR on Proving in telling the complete story of each group's work.

Group 1 Assessment via the CPR on Proving

Group 1 showed a strong collaborative effort in proving the given statement but did not reach a complete proof within the given time. Figure 2 below provides an aggregation of the coding for Group 1's proving attempt in the 25-minute span. I provide an explanation for the coding in each subcategory below.

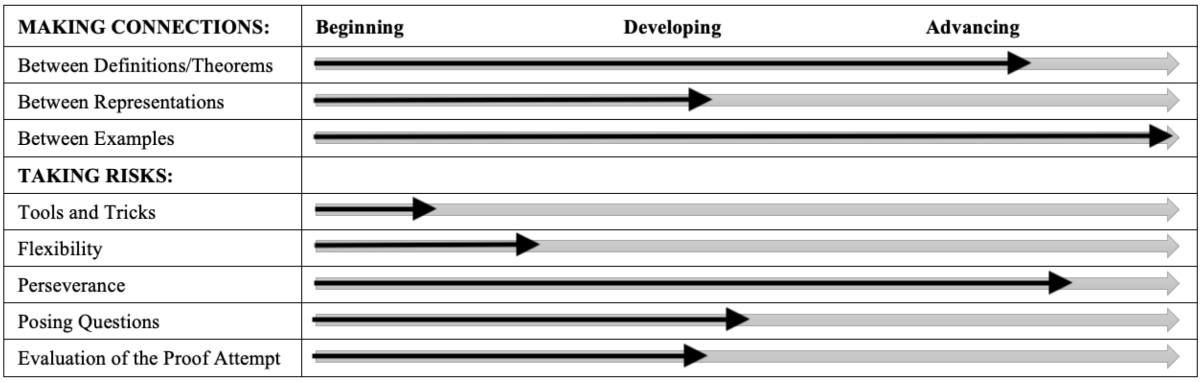

Figure 2. Levels of Group 1's Work

<u>Between Definitions/Theorems</u>

Group 1 demonstrated making connections between a variety of definitions and theorems. The group explicitly discussed the definitions of prime number, twin primes, natural numbers, and subsets. They connected their work to facts like the product of two negative numbers must be positive, and the idea that if a theorem holds for a larger group then it must apply to the subset. The group implemented the definitions of subset, natural numbers, prime, and twin primes in their proof attempt and made connections to the proof of a similar theorem (the product of two negative numbers must be positive), so I coded their work as "advancing" in the definitions/theorems subcategory.

<u>Between Representations</u>

Throughout their proof attempt, the group used an algebraic representation for two numbers differing by two and to represent a perfect square:

> *Matthew:* Yeah, because if we look at it and n squared is a perfect square, we have n minus 1 times n plus 1, which would be our two prime numbers here, FOIL it out, and we get n squared minus one.

Nathan attempted to formulate an alternative representation by describing the number in-between the twin primes as the average of the two numbers, but Nathan could never accurately verbalize the idea, and Matthew and Danvir did not seem to make a connection from Nathan's idea to their original representation:

> *Nathan:* We found two ways to do it. If you could do n squared minus one, and that will just give you the numbers or if you take any two numbers that are separated by two and then average them, divided by two minus one that will also get you the right number.

Group 1 also attempted to use symbolic representation by looking up appropriate symbols for "element of" and "subset" to use in their proof; the group, however, did not seem to make use of a conceptual connection between the symbols and their other work. Since Group 1 attempted different representations and made some incomplete connections, I coded their work as "developing" in the representation subcategory.

<u>Between Examples</u>

Group 1 began their work on this proof by gathering examples of twin primes and testing that their product is one less than a perfect square. The group eventually noticed that the integer in between twin primes is the one being squared:

> *Matthew*: I feel like we shouldn't be focusing on the prime numbers because I'm looking at these examples that we have down here, and the thing that's being squared is the number that's in between them. Like five and seven, that's the one that's one off from 36.

This realization led Matthew to develop the algebraic representation mentioned above in the "Between Representations" section. Eventually Matthew questioned what might happen if the numbers differing by two were not prime numbers, and Nathan tried examples in his calculator and conjectured that the statement was also true for other numbers differing by two. This eventually led Matthew to test examples of negative integers and non-integer values. After reviewing the definition of prime numbers, the group stopped pursuing new examples and further generalization because they realized they only needed to prove the statement for natural numbers to accomplish their given task.

Group 1 used several specific examples and synthesized information from their examples to construct their proof. I assessed the group's work as a high level of "advancing" in the examples subcategory.

<u>Tools and Tricks</u>

Group 1 used minimal tools and tricks in their proof. The group used the distributive property, or "FOILing," which stands for "first, outer, inner, last," to multiply $(n-1)(n+1)$. The group, with guidance from the instructor, also discovered and utilized the tool of searching for a more general statement which is simpler to prove. For this reason, I labelled the group's attempt at the "beginning" level for the tools and tricks subcategory.

<u>Flexibility</u>

In the attempt of their proof, Group 1 followed a single proving technique. Early in their attempt, Danvir challenged the notion of proving through examples and encouraged the group to pursue a more systematic approach but struggled with how to formalize the twin primes:

> *Danvir*: Is it fair to say that the best approach is just to go through examples? … I'm basically saying that. So, I was doing an example yesterday, trying to prove negative times negative is positive. Within that, there is no… you can do it better by not doing it through examples, you can actually take different properties and try to explain why

> that's the case. And I think this problem, it doesn't seem to resonate that that approach works. It's like here, you have to first define the twin prime set. There's no mathematical way to calculate …

After Danvir's comment, the group continued generating examples to eventually notice a pattern and formulate an algebraic representation. Since only one true proving strategy was employed, but group members suggested possible other directions that were not pursued, I labelled the group's work in between "beginning" and "developing" for the flexibility subcategory.

<u>Perseverance</u>
In my coding process, I found few moments suitable to code specifically for perseverance; however, after looking at the group's proof attempt as a whole, I concluded the group demonstrated perseverance at an "advancing" level. The group worked diligently on the proof the entire time allowed during class and continued to pursue further generalizations and push the boundaries of the statement.

<u>Posing Questions</u>
Most of the questions posed by Group 1 regarded the generalization of the given statement. For example:

> Nathan: Nine's not a prime, but it still works for that. Does it work for every example like that? And if so, they're all expressed by that formula.

Such questions clarify the given statement. The group also posed questions clarifying definitions of sets such as the natural numbers. For this reason, I assessed Group 1's work as "developing" in the posing questions subcategory.

<u>Evaluation of the Proof Attempt</u>
There were a few moments where the group reflected upon a successful or unsuccessful proof attempt. For example, Danvir offered a critical opinion of proving through examples only, and Nathan and Danvir recognized the success of Matthew's algebraic representation of the statement. Because of these instances, I placed the group at the "developing" level of evaluating their proof attempt.

Group 2 Assessment via the CPR on Proving

In contrast with Group 1, Group 2 displayed much less collaboration; nevertheless, the group was able to formulate a proof sketch to present to the rest of their classmates. Figure 3 below provides a summary of the coding for Group 2's proving attempts in the 25-minute group work session. I provide an explanation for the coding in each subcategory below first for the Making Connections subcategories and then the Taking Risks subcategories.

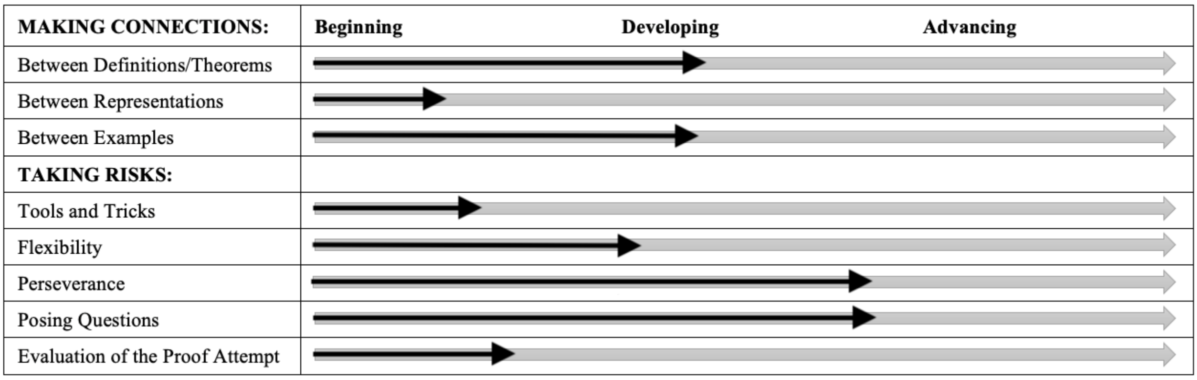

Figure 3. Levels of Group 2's Work

<u>Between Definitions/Theorems</u>

Group 2 discussed and/or used the definitions of: twin primes, consecutive, Pythagorean triples, absolute value, and subset. Susan noticed early in the group's process that the example of three and five (with four as the corresponding number squared) created a Pythagorean triple, and she proceeded to attempt to make a connection (without success) from twin primes to Pythagorean triples on her own throughout the group's time. Mason eventually used the definition of a subset to inform his ideas on the proof. Because the group identified relevant definitions and attempted to connect them to their proof but did not successfully implement them, I assessed Group 2 as "developing" in the definitions/theorems subcategory.

<u>Between Representations</u>

Group 2 attempted to rewrite the given statement in more formalized mathematical language, but the proof attempt itself employed only one representation, an algebraic representation. Susan rephrased the given statement at the beginning of the group's work, but she made no further attempt to represent her rephrasing with an alternate representation:

> *Susan:* Ok so perfect square minus one equals twin prime one times twin prime two.

Mason introduced the use of an indeterminate "n" to approach the proof:

> *Mason*: I set up an equation like with the lowest number being n and then n plus one is on the left side, that's -- that number is being squared, because it's the number in the middle and then being subtracted by one because that's one less than the perfect square. And then on the right side, I'm multiplying that lower number by the one two above it. And it seemed like I was getting to, after algebraic manipulation like two equal particles. It seemed that way.

Because Group 2 provided only one representation and did not attempt to connect it to other representation, I coded their attempt as "beginning" in the representations subcategory.

<u>Between Examples</u>

In Group 2's proof attempt, they generated a few examples of twin primes and Mason tested the statement using two examples. Mason, when verifying that the statement works for 11 and 13, noticed that the number being squared is 12, the number in between 11 and 13. Making this connection eventually led Mason to formulate his algebraic representation. Because the group made this connection, I assessed the group as "developing" in the examples subcategory.

<u>Tools and Tricks</u>

Group 2, like Group 1, used the distributive property to multiply $(n)(n+2)$ and $(n-1)(n+1)$. This group also discovered the tool of searching for a more general statement which is simpler to prove. Because the tools used in the proof were conventional for students at this level, I placed the group's attempt at the "beginning" level for the tools and tricks subcategory.

Flexibility

This group pursued two directions in this proof. Susan proposed exploring the relationship between twin primes and Pythagorean triples, and Mason led the group in an algebraic proof. While the group did acknowledge two different techniques, Susan displayed lack of flexibility in accepting that Mason's algebraic proof of a more general statement worked for prime numbers in this conversation:

> *Mason*: I did n plus one squared minus one equals n times n plus two.
> *Susan*: But it has to be a prime number.
> *Mason*: Yeah, but it's true for all numbers
> *Susan*: Right to be...to solve.
> *Mason*: You see what I'm saying?
> *Susan*: Yeah, it works for any number, but we're talking about prime numbers.
> *Mason*: That would fall into that group.

I am looking at this group as a unit, so in assessing their proof attempt, I placed their work at the low "advancing" level because they pursued two directions, but I did not place them higher because the two approaches were not collaborative but simply two distinct approaches of two individuals.

Perseverance

It was difficult to evaluate this group's perseverance because each student's investment level differed. Susan struggled to engage with Mason's algebraic representation, and Craig did not seem willing or able to contribute meaningfully to the group aside from rewriting the statement using mathematical symbols. If I were evaluating the students individually, I would place Mason at a low "advancing" level, Craig at a "beginning" level, and Susan at a high "beginning" level. In order to assess the group as a whole, I believe the lack of perseverance from Susan and Craig weighed more heavily on the group's attitude, so I placed Group 2 at a lower "developing" level in the perseverance subcategory.

Posing Questions

Throughout their collaboration, the members of Group 2 posed a question regarding the connection between twin primes and Pythagorean triples, the definition of prime numbers, and the possibility of the statement being generalizable to all numbers. Because of the nature of these questions, I coded Group 2 between the "developing" and "advancing" levels in the posing questions subcategory.

Evaluation of the Proof Attempt

In Group 2's work, there were only a few instances of the group reflecting upon their attempts. Mason acknowledged Susan's observation about Pythagorean triples and suggested that the group was "getting somewhere." Susan complimented Craig's ability to write the statement using mathematical language and symbols, and frequently complemented the group on employing "alternative representations." The group (according to the CPR), however, did not employ alternative representations. Group 2's evaluations of their proof attempts were mostly local and/or inaccurate. Thus, I located Group 2's attempt at a "beginning" level for this subcategory.

Effectiveness of the CPR on Proving for Collaborative Work

The Creativity-In-Progress Rubric on Proving was designed to describe the creativity displayed in an individual's proving attempt and to facilitate personal reflection, yet in this study I have attempted to describe a group of students as a unit using the CPR. When reviewing the CPR on Proving assessment of Group 1, there are only a few moments neglected in consideration of this group's holistic story; the CPR on Proving, however, was more difficult to implement for Group 2.

Group 1 worked very well together and constructed their proof by challenging, recognizing, and building upon the ideas of their fellow group members. This is not represented well by the CPR on Proving. Danvir first challenged the group by proposing that simply generating examples would not be enough to prove the statement and suggested the use of more general principles. Nathan generated and verified numerous examples. Matthew noticed through the group's list of examples that the perfect square referenced in the statement is the result of squaring the "middle" number of the twin primes used, and he wrote the algebraic representation of the statement: $n^2 - 1 = (n - 1)(n + 1)$. Danvir and Matthew argued over how to restrict this equation to only be true for twin prime numbers, and Matthew questioned what would happen if they plugged in a non-prime. Prompted by this, Nathan tested an example using $n = 10$. The three young men proceeded to collectively generalize the statement. The coaction of Group 1 is not evident by viewing the CPR on Proving assessment of their work. Intersubjectivity allows for the observation that each student had his own ideas about how to approach and carry out the proof, yet they listened to one another and reacted in a way that was productive and facilitated collaborative creativity. Each student's individual creative contributions to the proof were deeply intertwined with their teammates, and this should be represented in a rubric of their creative proving process.

In contrast to Group 1, Group 2 did not construct their ideas collaboratively. I imagine the CPR on Proving would have been easier to implement for each individual group member than the group as a unit. There was ample time in the video where the students silently worked on their own papers. After her observation about Pythagorean triples, Susan worked independently on trying to make a connection to twin primes. Craig was focused on using mathematical symbols and language to rewrite the statement and incorporate Mason's realization about the perfect square being generated by the number in between the twin primes. Mason's ideas were the foundation of Group 2's proof. From what is seen in the video, Mason is the only group member who tested examples in the statement; Susan and Craig only generated examples of twin primes. Mason's engagement with the examples allowed him to make the connection to the number in the "middle" of the twin primes being the square root of the perfect square mentioned in the statement. This realization led Mason to represent the statement algebraically with the equation: $n(n + 2) = (n + 1)^2 - 1$. Mason finally made the conclusion that his equation would work for any number, and hence it would prove the statement for twin primes. Susan challenged this approach, and while she was eventually able to revoice Mason's ideas, she remained confused about the proof at the end of the group's time together:

> *Susan*: I can be emotional support this time, but I'm not explaining this.
>
> *Susan* [2 minutes later]: You [Craig] did the specified that's on the board. And then we had to generalize but you still need to connect and prove that this still fits within the

> generalized, because you know, you always have the math equation, it's like, for all numbers except one. So like, that's to make sure that this doesn't apply.

Later in the class, Mason presented the group's work to their classmates. Intersubjectivity in this holistic story of Group 2's interaction explains how the different ideas and values from the group members shaped the eventual proof product, composed primarily of Mason's ideas. These group members did not continually respond to or react to the contributions of their peers to create something together, but Mason's confidence and willingness to lead the group dominated the direction of their work. If this group had been an improvisational theatre troupe, all three actors were telling a different story on stage; Mason directed the audience's attention to the story he wanted to tell, and Susan remained confused by the plot created by the troupe at the end of the performance.

Discussion

Modifications to the CPR on Proving for Collaboration

After assessing the effectiveness in the CPR's ability to tell a holistic story of creativity in collaborative proving, I have developed three suggestions for modification to the CPR on Proving when applied to a group as a unit. The CPR on Proving has been used mostly for formative assessment and student reflection. My suggestions below are motivated by improving the rubric to enhance its use in revealing mathematical creative abilities in tertiary students and helping students overcome common roadblocks in proving by reflecting upon their creative process in a group setting.

First, I suggest an additional category on collaboration. The addition of this category to the rubric will require reflection upon the group's ability to recognize and build upon one another's contributions. In the continuum of evaluating collaboration, at the beginning level the group members' attempts of the proof are individual. At the developing level, group members acknowledge one another's ideas and attempt to produce a cohesive proof. Finally, at the advancing level, the group constructs a proof collaboratively and successfully incorporates the contributions of each group member cohesively. For example, in the cases examined in this paper, Group 1 would be placed at the "advancing" level, while Group 2 would be placed at a high "beginning" level.

The second suggestion for modification of the CPR on Proving is the expanding of the "posing questions" subcategory. I observed in my analysis that students working in a group frequently posed suggestions or conjectures where they may have been posed as questions in an individual setting. I would expand this category to be "posing questions/making suggestions." For example, in Group 1's attempt to generalize the statement, Matthew said:

> It's a positive number. I'm going to say positive because I don't think it would work with negative... But I don't know, because there's only... Ugh! If we include negatives, then it could be like one and zero. So maybe we can say any nonzero number.

This suggestion prompted a conversation between Matthew and Nathan regarding whether or not the statement would apply to negative numbers similar to how the conversation could have proceeded if Matthew had asked "can we include negative numbers?" The CPR on Proving did not capture this moment because Matthew did not phrase his thought as a question even though his body language and tone indicated he was seeking the groups input on his suggestion.

Finally, when implementing the CPR on Proving in a collaborative setting as a reflective tool, I suggest that students evaluate the proof attempt by first viewing the group as a unit and then using the CPR on Proving to assess their individual contribution to the group. For example, in Group 1, Nathan contributed a majority of the examples, Danvir was strong in making connections to other theorems and flexibility, and Matthew posed important questions to the group. In Group 2, Susan and Criag would have had lower scores overall in most categories, while Mason would have had higher scores in most categories. Encouraging group members to examine their contribution will allow them to notice areas for improvement, appreciate contributions of their group members, and learn how to approach proofs more collaboratively.

Limitations of the Study

This study used only video data to analyze the collaborative proving process. The original presentation of the CPR on Proving utilized LiveScribe pens (Savic et al., 2017) to capture both audio and written work. This study may have been improved by the analysis of written work of each group member, particularly in the examples and representations subcategories. It was apparent in the videos that students were writing on their own papers throughout the group discussion, and access to this data could have provided further insight into the groups' performances. It is possible that analysis of the student written work may have enlightened other suggestions for modification to the CPR on Proving.

Further, this study is limited in its ability to conceptualize creativity as relative to the students' backgrounds and abilities. When the CPR is implemented as a reflective tool for students, they can assess themselves relative to their own experiences and perceptions; as a third-party evaluator in this study, however, I had to make assumptions about the participants' knowledge and backgrounds to determine how novel or routine the strategies or techniques used were for the students.

Future Research

Mathematical creativity is a central tenet of tertiary mathematics education, especially in proof. For this reason, future research should focus on determining how to foster creativity in tertiary mathematics students. There are many pedagogical strategies thought to foster creativity. For example, problem posing (Silver, 1997), model-eliciting activities and phenomenon-based learning (Asahid & Lomibao, 2020), and multiple solution tasks (Leikin & Elgraby, 2020) have been conjectured to improve mathematical creativity skills. A clear direction for future research stemming from this study is to examine if collaboration in proving fosters mathematical creativity. Does a group develop more creative ideas than its members could independently?

The CPR on Proving has offered a wonderful contribution to the world of undergraduate mathematical creativity in proof by providing a reflective and formative assessment tool to encourage and emphasize creativity. In my experience, as students progress in their mathematics education, they increasingly look to their peers for support and engage in collaboration. Since these suggested modifications were made, I have modified the CPR for use in groups and implemented the CPR with student groups in an Introduction-to-Proofs course. Preliminary results of this study show that student groups may be more or less reflective when using the CPR in their groups and that the CPR can provide opportunities for students to point out ways in which their peers contributed to their proving process that were previously unnoticed. Analysis from this project is ongoing, but I refer the reader to Heath et al. (2022) for an overview of preliminary results.

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