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Constructivist Learning Design: A Platform for Differentiated Instruction towards Mathematical Literacy

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In this paper, I present the Constructivist Learning Design (CLD) as an instructional design for teaching through problem solving and describe how differentiated instruction complements CLD to engage diverse learners in the mathematisation process for construction of knowledge towards mathematical literacy. An example of a CLD task implemented with Secondary Two students (aged 13-14) in Singapore schools as part of a research study is discussed in view of the mathematisation process elicited through use of differentiated instruction. We argue that CLD has the potential to support teachers in planning for deep learning.

Keywords: Constructivist, Deep Learning, Differentiated Instruction, Mathematisation, Secondary Schools

Introduction

Mathematics curricula around the world have outlined how mathematical literacy, sometimes referred to as "quantitative literacy" or "numeracy", is interpreted in different educational systems (Geiger et al., 2015). The notion of mathematical literacy has evolved over the years. In PISA 2022 mathematics framework, the Organisation for Economic Co-Operation and Development (OECD) emphasizes the importance of mathematical reasoning to solve problems situated in the real-world (see section on mathematical literacy (OECD,2022)). This is evident in the first half of their definition of mathematical literacy: "an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts, and tools to describe, explain, and predict phenomena". OECD further outlines the purpose of mathematical literacy in the second half of their definition: mathematical literacy "helps individuals to know the role that mathematics plays in the world and make the well-founded judgements and decisions needed by constructive, engaged and reflective 21st Century citizens". Underlying this definition is whether an individual can draw upon appropriate knowledge, skills, and strategies to formulate procedures or structures to represent a real-world phenomenon mathematically. Proper reasoning and assumption-making within the context is necessary. A mathematical representation which had originally evolved from a specific real-world context can be further refined or abstracted for its applicability and generalisability across other situations for purposeful decision-making. Hence, also implicit in this definition is for mathematics curricula to help students appreciate the relevance of mathematics in the 21st century in making sound judgements for day-to-day living as they contribute actively to society in meaningful, thoughtful, and responsible ways.

The vision of mathematical literacy presented by OCED challenges one's narrow interpretation of the utilitarian view of mathematics. Preparing students to be mathematically literate goes beyond equipping them with knowledge and skills and providing them with opportunities to engage with different types of mathematical problems. As the proponents for the inclusion of real-world problems in mathematics curricula (Gravemeijer, 1994; Ng, 2011, 2018; Stillman et al., 2016; van den Heuvel-Panhuizen, 1999) have argued, it is crucial for students to learn how to mathematise during problem-solving. It is through the mathematisation process (de Lange, 2006) where students formulate, represent, solve, interpret, and reflect upon their solution for a mathematically informed decision within the constraints of the real-world context that one's mathematical literacy can be enhanced (Widjaja, 2011). Problems arising from real-world situations are often open to multiple interpretations and several solution pathways. These can provide platforms for intuitive mathematical reasoning and processes to evolve which may later be formally represented, resulting in more insightful mathematical learning (English, 2006).

Teaching through problem solving (Schroeder & Lester, 1989) is one way that teachers can conceptualise students' learning journey towards mathematical literacy. Although not a necessary condition for teaching through problem solving, the affordances of mathematical activities, tasks, or problems situated in real-world contexts that are open for multiple interpretations allow for mathematisation processes to be elicited from students for richer, meaningful construction of knowledge. During teaching through problem solving, students either work on the problem individually or in groups before the mathematical object or concept is formally introduced or taught. Thus, the problem is carefully designed to encourage varied representations during the knowledge construction process. The teacher becomes the facilitator when students are exploring about the problem, focusing on observing and listening before asking questions to help students clarify, explain, and reflect on their thinking. At times, students can be prompted to use written representations of their mathematical reasoning and encouraged to revise their arguments. After an independent exploration phase, the teacher then gathers the class for sharing and discussion. Here, different solution pathways are examined. Where appropriate, the teacher draws out generalisations as the solutions are compared, and highlights key points associated with reasoning that are productive in the co-construction of a mathematical object or concept, setting the stage for a more formal introduction or direct instruction. In teaching through problem solving, teachers encourage students to be independent learners (Takahashi, 2017) who can be creative problem solvers, opening their minds for varied solution pathways (Lim, 2020).

An instructional design which realises teaching through problem solving in Singapore schools is Constructivist Learning Design (CLD) (Lee et al., 2021). Mathematical problem solving has been the main goal of the Singapore school mathematics curriculum framework (MOE, 2020, p. 9) for many years. Singapore students are encouraged to solve problems presented as straightforward routine tasks, or complex, non-routine challenging tasks. The problems can be situated in a variety of real-world contexts (e.g., everyday life, future work, societal and global), and can be open-ended, or ill-defined. It is thus important for students to use mathematical processes such as abstracting, representing, and modelling (p. 10). Particularly for a real-world problem, students begin mathematising when they formulate the problem mathematically by understanding the problem context, making reasonable assumptions, simplifying the problem conditions, and representing the problem mathematically (p.10). This is followed by appropriate choice and use of tools and strategies when solving the problem. The solution

(1977), internal cognitive conflicts stimulate knowledge construction because in the process of resolving mental disequilibrium, the learner engages actively in examining existing schemata for congruence, gaps, and misalignment of the newly constructed knowledge or concept. Whether and how the concept is learnt and understood depends on which internal cognitive process is being activated: assimilation or accommodation. Assimilation occurs when new information can be integrated into existing schemata. Accommodation takes place when the learner reorganises and perhaps restructures his or her schemata to attend to the cognitive conflict brought about by the new knowledge. Lee et al. (2021) called for the teacher's instructional choice to include various problem situations which may present impasse, failure, ambiguity, or uncertainty, and for facilitation and support from knowledgeable others (e.g., teacher, peers) to build upon the original ideas, strategies, and preconceptions shared by students (p. 326). In doing so, the learner's efforts are acknowledged while being prompted to work towards a more sophisticated or efficient pathway after assessing the initial solution for its affordances and limitations in terms of applicability and generalisability. This may bridge the gap in the learner's Zone of Proximal Development (Vygotsky, 1978) more meaningfully.

The third and final constructivist principle that Lee et al. (2021) referred to is the importance of learners' evaluation of the "viability of individual understandings" through social negotiation of meaning. Lee and his team recognize that "knowledge is mutually built when learners both refine their own meanings, and help others find meaning" (p. 326). The three principles Lee's team referred to are also in tandem with Wong (1994), who was one of the first to examine the possibility of incorporating constructivist philosophy in mathematics instruction within Singapore schools. Indeed, as Ernest (1991) aptly put it more than 30 years ago, mathematics is a "living social construction, with its own value, institutions, and relationship with society in the large (p. 107). This is more than verified today with the emergence of 4IR and the latest OCED's definition of mathematical literacy.

CLD as an instructional design

Drawing upon the above, Lee et al. (2021, p. 327) adopted four guiding principles in creating CLD as an instructional design. It should:

- afford the elicitation and building upon of students' pre-existing understandings of a subject matter;
- aid in the development of an organised and interconnected knowledge that facilitate retrieval and application;
- engage students' thinking about their thinking and learning through cognitive disequilibrium, the realisation of one's potential and reflecting on the affordances and constraints of their solutions; and
- build a social surround that allows for interpersonal and social nature of learning.

In addition, Lee and his team also referred to three effective learning designs with varying degrees of success in Asian classrooms: Productive Failure (see Kapur, 2008, 2010), the Open-Ended Approach (Becker & Shimada, 1997), and the Post-Tea House Teaching Approach (Tan, 2013) when conceptualising the instructional structure of CLD. Fundamental to the three learning designs mentioned above are two phases in organising a lesson or a sequence of consecutive lessons to attain a learning goal. The two phases are developmental. During the first phase, students activate their prior knowledge and experiences to respond to carefully designed problems, tasks, or activities aimed at eliciting a variety of knowledge structures during the construction of concept, and "Representations and Solution Methods" (RSMs;

of the concept or mathematical object, factoring in generalisability for all cases, or identifying the parameters of applicability for each case.

Next, we explain the two sequential phases in CLD. The problem-solving phase presents students with a task consisting of a problem with the mathematical concept to be constructed embedded in its design. Students are first encouraged to mathematise about the task individually and to devise as many RSMs as possible to respond to the task. These RSMs may well be the initial, intuitive constructions of the concept. When the students proceed to work with their peers, more complex RSMs may arise because of interactions between different perspectives from a variety of entry points as the students draw upon their existing schemata. RSMs are typically recorded in writing with or without diagrams so that teachers can analyse them ahead of time before the next phase is enacted. During analysis, the RSMs are classified and ranked according to their potentials and limitations in addressing the critical features of the concept which the context of the task serves to inspire. In the instruction phase, the teacher displays the different clusters of RSMs and engages students in a series of discussions, in groups or as a class. Such discussions aim to surface the critical features of the concept outlined by the RSMs and compare across RSM clusters to analyse and reflect on the potentials and limitations of the RSMs in representing all possible cases or in addressing different cases based on identified conditions. The RSMs can be revised subsequently as students are prompted for further mathematisation towards deliberate goals to create more sophisticated RSMs from their initial ones, building upon each other's RSMs. Depending on how closely aligned are the subsequent RSMs to the desired canonical solution, teacher facilitation can request for class contributions to confirm the canonical solution, or to continue the mathematisation processes with a sharpened focus towards a canonical solution. Deep learning of the concept may result from this. CLD contrasts with direct instruction because students are provided with opportunities to construct knowledge about the concept before the concept is formally introduced in its abstract representation. We propose that CLD is one of many approaches (including direct teaching) a teacher can adopt in the mathematics classroom. Once a canonical solution is reached during the instruction phase, the teacher may embark on direct teaching for further explication and application of the canonical solution in different examples.

Lastly, we identify four predominant student-actions during the two phases of CLD: explore-generate, compare-connect. These are accompanied by arrows moving out of the phases to illustrate the phase in which the student-actions are likely to occur. Explore-generate are actions undertaken by students when they work on various scenarios, cases, and examples of events articulated in the problem. They can generate a variety of mathematical representations, arguments when they come up with initial RSMs. Compare-connect are actions which students may naturally display when they analyse and compare between RSMs during discussions. Connections can be drawn between ideas, between mathematical cases, and between prior understandings as students examine the similarities and differences amongst the RSMs. The double arrow between explore-generate and compare-connect highlights the cyclical nature of these processes during the development and refinement of RSMs.

Interacting elements considered in the design of CLD tasks

The affordances of CLD tasks are built-in, based on purposeful design considerations. Task design is crucial in CLD because "theory and its design tools are viewed as a front-end resource" (Kieran et al., 2015, p. 29) for teaching through problem solving. Four interacting elements are involved when crafting CLD tasks. These are as follows: (a) the task has an open-

student's existing schemata as a RSM is examined particularly for gaps, limitations in representing cases, or contradiction in logic (Ng, et al., 2021a). The next section presents one CLD task and the RSMs generated by Singapore students who attempted the task so that we can examine the mathematisation processes in generating the RSMs and illustrate how the teacher can employ DI towards mathematical literacy accommodating different ways of formulating, employing, and interpreting mathematics to solve the problem. All CLD tasks were implemented in participating Singapore secondary schools. More examples of CLD tasks, discussions on authentic student produced RSMs, as well as implemented and suggested teacher facilitation moves can be found in Ng et al. (2021a).

An Example of CLD Task: Simultaneous Linear Equations

Solving simultaneous linear equations (typically one pair of such equations) is taught in Secondary Two mathematics classrooms (students aged 14) in Singapore schools. Prior to this topic, students have learnt to solve linear equations in one variable and to draw and interpret graphs of linear equations, including the concept of gradient. Figure 3 shows the open-ended task situated in a context familiar to many teenagers. Students have yet to learn about what solving simultaneous linear equations entails when this task was implemented in the problem-solving phase in CLD. For the Secondary Two students, solving simultaneous linear equations requires an understanding of both graphical and algebraic representations of linear equations and how these are connected. One of the following three approaches can be adopted to solve simultaneous equations: (a) the algebraic method where the exact value of the solution (if it exists) of a pair of simultaneous linear equations can be found through strategies (e.g., substitution, elimination), (b) the graphical method where there is visualization of the behaviours of pairs of values which satisfy each given equation, and it is possible to locate an exact pair of values signalling common point satisfying both equations simultaneously (if this point exists), and (c) a combination of algebraic and graphical methods which complement each other particularly when solving complex problems (e.g., more than two linear equations are involved) (Ng et al., 2021a).

Several critical features of simultaneous linear equations are embedded in the task presented in Figure 3 (Ng et al., 2021a, p. 74). Firstly, a solution to the problem involves finding pairs of values which satisfy at least 2 linear equations simultaneously. When interpreted in the context of the given problem, students determine what it means to identify the various pairs of values (there are three linear equations in the problem), and how this can help them make informed choices in response to the task (i.e., identify the cheapest service corresponding to the different sizes of sound files). Secondly, if using the graphical method, a pair of values which satisfies two linear equations simultaneously is given by the coordinates of the point of intersection (x, y) of the graphs representing the two equations. Thirdly, the graphs of two (non-parallel) linear equations with different gradients will signal a reverse order of higher and lower charges beyond a certain sound file size after the point of intersection between a pair of graphs. Fourthly, when using the algebraic method, the two quantities (sound file size and monthly charges) are not only interpreted from the data given in the problem but are also represented by common variables across the three linear equations formed in the process. The charge incurred is a function of the sound file size determined by a fixed ratio (i.e., the gradient of the linear graph is constant). The various RSMs produced by students for the task are clustered according to their potentials and limitations in the mathematisation process towards the canonical solution. We select three RSM clusters to illustrate teacher facilitation for DI below.

Music on the Go

Albert, who likes to listen music on the go, would like to subscribe to a music download service. He is interested in 3 high quality music download services - *Apple Tunes, Bandora, and Cloud Sounds* – and found an advertisement of their monthly charges. Table 1 below shows the charges.

Charges	Music Download Service		
	Apple Tunes	**Bandora**	**Cloud Sounds**
Monthly Membership Fee	**$ 0.00**	**$ 4.00**	**$ 20.00**
Monthly charge per GB of sound files downloaded	**$2.70**	**$2.00**	**$1.00**

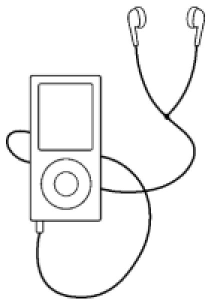
Table 1: Charges of 3 music download services. All services charge for membership and sound files downloaded monthly.

Based on the advertisement, Albert would like to know which service provider is the ***cheapest*** for ***different*** sizes of sound files downloaded. He is not sure how this can be done, and therefore asks your group to help him out on this. Here is what you and your group mates must do:

(i) compare among the 3 music download services, and work out which service would be the cheapest for different sizes of sound files downloaded;

(ii) use as many methods as possible when making your comparisons; and

(iii) for each method and solution, indicate as much as possible, the exact values of the sizes of the GB of sound files and their costs.



All the best, and remember, don't give up until you have come up with as many methods as possible!

Picture downloaded from http://clipart-library.com

Figure 3. Simultaneous Linear Equations Task (Taken from (Ng et al., 2021a). Reprinted with permission.)

Cluster 1 RSMs: trial and error

An example of an RSM belonging to this cluster is presented in Figure 4a. Such RSM is created using trial and error where random values of sound file sizes are tested. Here, the student interpreted the relationship between monthly charges and sound file sizes according to the given data in the table. The mathematization process is not completely represented in written form within the solution. The solution does not articulate the linear relationship between the two variables. The student might have formulated linear equations to work out the values as seen in the figure drawing upon prior knowledge of linear graphs but there is no further evidence that he or she managed to move on from this preliminary conception towards a simultaneous solution involving at least two of the equations. There was also no further investigation about the exact size of the sound file, beyond which, one service will start to charge more than the other. It appears that the RSM did not take into consideration the critical features of simultaneous linear equations. When adopting DI to address the needs of this student, the teacher may ask the following questions (Ng et al., 2021a, p. 75) based on what the student has offered in the solution, to prompt for more explicit mathematical reasoning, and to channel the student back to explore-generate actions:

- Can you explain the relationship between the size of the sound files and the download charges?
- How will the charges differ between the service providers when you download different sound file sizes?
- You concluded that Apple Tunes is the cheapest service provider. Do you think it will always be the cheapest for all sizes of sound files downloaded? Why?

It is understandable that students may produce cluster 1 RSMs because they have yet to be formally taught how to use the algebraic method (i.e., by substitution or elimination) to solve a pair of simultaneous linear equations. Here, the purpose of teacher facilitation is to target at deep learning about the relationship between the two variables. Further to this RSM, students may well adopt a series of strategic "guess-and-check" comparisons to narrow down to a possible cheapest service provider. This improved RSM would be useful during the instructional phase of CLD where the teacher compares and discusses the different RSM clusters, particularly contrasting and connecting the algebraic and graphical methods.

Solution A

Suppose the downloaded sound file size is 5 GB.
Apple Tunes charges $13.50.
Bandora charges $14.
Cloud Sounds charges $25.
It can be concluded that **Apple Tunes** is the cheapest.

Figure 4a. Cluster 1 RSM (Taken from (Ng et al., 2021a). Reprinted with permission.)

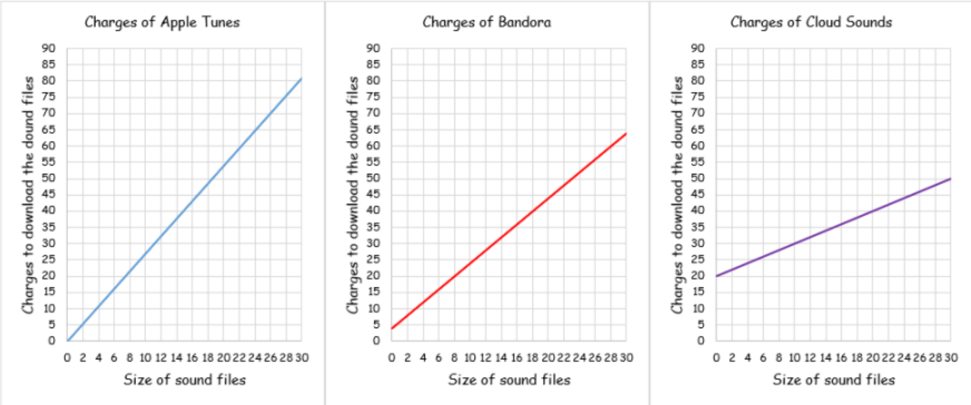
Cluster 2 RSM: drawing the graphs separately

An example of an RSM belonging to this cluster is presented in Figure 4b. Here, the student drew each of the three graphs separately and attempted to compare them using his or her prior understanding of gradient. The student has chosen to formulate the problem graphically and has employed appropriate techniques to do this. However, the graphical comparison between the charges of the three service providers could be enhanced by teacher prompts during DI targeting at locating a common point on different pairs of graphs to see a reverse order of higher and lower charges beyond a certain sound file size. In addition, the student can also be prompted to interpret and make sense of why we need (at least) a pair of values which satisfies two linear equations simultaneously (before moving on to three graphs with one another) and how this can be obtained from the coordinates of the point of intersection (x, y) of the graphs representing the two equations. Two possible teacher questions can be as follows:

- You have done three separate graphs. How did you compare between the three graphs? What were you looking out for when you were comparing between the graphs in order to answer the question?
- How can you make the comparison of all three graphs more efficient to see at which sound file size will a particular service provider be cheaper?

Solution C

The graphs below show the relationship between the size of the downloaded sound files and the charges by each of the service providers.



From the graphs, the charges of Apple Tunes increase sharply while the charges of Cloud Sounds increase slowly. So, Apple Tunes is the cheapest when size of sound files downloaded is small, while Cloud Sounds is the cheapest when size of sound files downloaded is big.

Figure 4b. Cluster 2 RSM (Taken from (Ng et al., 2021a). Reprinted with permission.)

Cluster 3 RSM: using graphical method to estimate

This RSM (Figure 4c) shows that the students could have recognised the linear relationships between the two variables across the three service providers. They appeared to have formulated the problem appropriately to explain the linear relationships between the variables. Their graphical representation provided useful visualisation for the gradients of two linear graphs, and the point of intersection marks how the charges differ after a particular sound file size. They were able to make sense of the context and why there was a need to identify the intersection point of the two graphs. From the short write-up they gave, they compared all three graphs, working with pairs of graphs one at a time. The RSM also encompass at least two critical features of simultaneous linear equations. The students were able to identify a pair of values which satisfies two linear equations simultaneously and this came from the coordinates of the point of intersection (x, y) of the graphs representing the two equations. In addition, from the students' explanation about their graphs, it seems that they were aware, at least between Apple Tunes and Bandora, that the latter service provider became cheaper if the downloaded file size goes beyond 5.5 GB. It was unclear, however, how their other paired graphical comparisons went. It would be important for them to note that the same axes and scale should be used when comparing more than one graph. Some of the teacher facilitation questions (Ng et al., 2021a, p. 78) catering to the needs of students here can be:

- What does the intersection point here show? Is this a sufficient response to the question posed in the task? Why?
- Is it possible to make comparisons between the three service provides on the same graph?
- What do you think will happen to the charges of the three services immediately after the intersection points of their graphs?

Solution D

We compared the charges to download sound files in pairs by **Apple Tunes** and **Bandora** using the graph below. The two lines intersect approximately at the point (5.5GB, $15). This means that the two services charge the same when the downloaded sound files' size is 5.5 GB. When the total downloaded sound files' size is less than 5.5 GB, the line representing **Apple Tunes'** charges is below **Bandora's**, showing that former is cheaper. **Bandora** is the cheaper one if the downloaded sound files' size is more than 5.5 GB.
We can make similar comparisons between **Apple Tunes** and **CloudSounds**, and between **Bandora** and **CloudSounds** graphically.

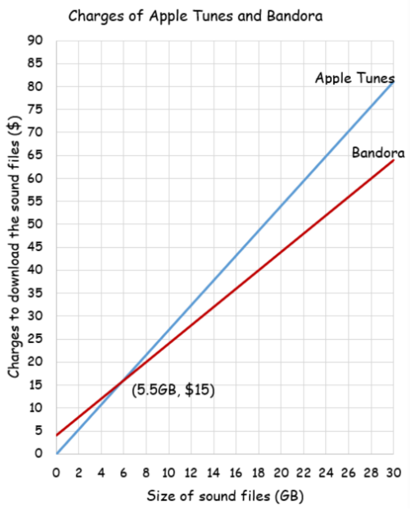


Figure 4c. Cluster 3 RSM (Taken from (Ng et al., 2021a). Reprinted with permission.)

Conclusion

In this paper, we present how the use of differentiated instruction complements CLD as an instructional design for an inclusive mathematics classroom. In CLD, diverse learners are provided with opportunities to mathematise open-ended problems situated in real-world contexts. Learners are encouraged to make sense of the situation as they engage in mathematical reasoning for knowledge construction. This is an important step towards mathematical literacy. Because of the iterative mathematising process supported by targeted teacher scaffolding in DI, RSMs are revised to address possible gaps and limitations. Deep learning can occur as knowledge constructed during the development and refinement of a RSM is examined repeatedly for soundness, generalisability, and applicability to other situations. A canonical solution is reached when it can explain, describe, or predict the real-world problem through an established mathematical relationship between quantities, the cases it can be applied to, and the conditions in which they can exist. Although Lee et al. (2021) reported that mathematics classrooms who adopted CLD had significantly higher scores for conceptual understanding, near, and far transfers of conceptual knowledge, work continues in CLD task development and trials in Singapore schools so that teachers can have an alternative instructional design with authentic resources and teacher facilitation suggestions to support them in working with students on challenging mathematical concepts.

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