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Connecting Current Mathematical Problem Solving Research Findings with Curriculum Proposals and Teaching Practices

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Mathematical problem solving is a prominent perspective for students to understand concepts and to apply mathematical knowledge. Indeed, problem formulation and looking for different ways to solve problems are crucial activities in the development of mathematics. Furthermore, understanding and characterizing what the process of posing and solving mathematical problems entails have been part of the research agenda in mathematics education. What are the important research findings in the problem solving domain and how can they be used in teaching practices? To address this question, three intertwined themes are discussed: seminal conceptual frameworks to characterize problem solving approaches; inquiring methods to problematize students' problem solving activities; and the influence of tasks or problems used in research to frame curricula and mathematical instruction. Throughout the presentation of these themes, teachers' and students' use of digital affordances is analyzed to discuss what ways of reasoning emerge in approaching and solving mathematical tasks. In this context, students' processes to transform the tools in instruments to explore concepts and to solve problems becomes essential for teachers to structure and implement problem solving activities.

Keywords: Mathematical problem solving; research findings, digital technologies; and problem solving learning environments.

Introduction and Background

A recurring theme in a problem solving approach to learn and construct mathematical knowledge is that students need to problematize their learning. This implies that they need to think of and conceptualize the study of the discipline as a set of dilemmas that need to be explained and solved through the consistent use of mathematical knowledge, resources, and problem solving strategies (Santos-Trigo, 2019). From this perspective, they pose and discuss questions to understand mathematical concepts or problem statements, design and implement a solution plan, and communicate and extend problem solutions. "Students should be allowed and encouraged to problematize what they study, to define problems that elicit their curiosities and sense-making skills" (Hiebert et al., 1996, p. 12).

Thus, students delve into concepts and problems through questions to explore meanings and identify objects' mathematical relations that are relevant to solve problems. Indeed, questions that students pose throughout all problem solving phases are relevant to identify material to understand concepts or examples and important to formulate and explore new problems (Cai and Hwang, 2023). In this process, students engage in a strategic plan (How should I deal and work on mathematical tasks? Who should I share and discuss mathematical tasks with? What online resources or platforms should I consult, etc.) and a tactical plan to implement a path and actions that lead them to understand concepts and to solve mathematical problems. Students' strategic and tactical plans also involve the activation and regular use of resources, tools, and online developments during the social confinement due to the COVID Pandemic, to work and discuss mathematical tasks with peers and their teachers. For instance, beyond the pandemic, communication apps have become important to extend mathematical discussions and to share ideas and problem solutions with peers and teachers beyond the regular class sessions.

In addition, students have started to consult online resources such as Wikipedia or digital platforms to review or extend their understanding of concepts, to analyze examples of problem solutions, or to work on available tests or online exams to assess and monitor their own knowledge and learning. In response to this, teachers need to set clear guidelines for their students when and in which way they could use online platforms and resources to work on mathematical tasks. In this way, teachers need to problematize the content that students are likely to consult exterior resources to understand mathematical concepts and to solve mathematical problems.

Furthermore, teacher's questions need to be relevant to frame and implement learning activities. Arcavi and Schoenfeld (2008) propose problematizing three related dimensions: The mathematical knowledge to teach, goals to achieve, and beliefs about teaching. In terms of the content, questions to discuss include:

> What is important about the mathematics, and why? … (What content, what processes? Is their approach conceptually oriented, procedural, etc.? What [perhaps implicit] messages does the teaching send about what it means to learn and do mathematics?), etc. (p. 285).

Similarly, teachers should reflect on teaching practices and posing questions regarding what their students can do and ways to participate in problem solving activities.

> What do teachers' classroom practices reveal about what they think their students are capable of learning (or should learn)? What does it say about their goals for students? What "lessons" (either intended or unintended) might students be learning from those classroom experiences? How are content and process goals seen, through the eyes of the students? (Arcavi and Schoenfeld, 2008, p. 285).

From this perspective, it is important to analyze the extent to which systematic use of digital technologies and online developments provide a set of affordances for learners to represent, explore, and understand mathematical concepts and to develop problem solving competencies. To this end, some themes and tasks that appear in the high school curriculum are discussed to illustrate the ways students can develop their reasoning using digital tools. In this paper, we illustrate the importance for students to rely on the affordance of GeoGebra to construct dynamic models of concepts in order to identify and explore mathematical relations that are relevant to solve problems in different domains including algebra, geometry, and calculus.

Conceptual Frameworks in Mathematical Problem Solving

What is essential in understanding concepts or how can the process of formulating and solving mathematical problems be explained or characterized? To what extent does a robust characterization of the problem solving process provide relevant information to structure and support students' learning environments? Questions such as these have been of interest to the mathematics, mathematics education, and teaching communities. Indeed, mathematicians, mathematics educators, and teachers have significantly contributed to understanding relevant features of the problem solving process, students' cognitive behaviors to work on mathematical tasks, and ways to structure curriculum and problem solving learning scenarios.

A salient issue is to recognize that a problem solving perspective involves the development of a way of thinking that reflects features of mathematical practices that go beyond finding and applying formulae and procedures to solve problems. It includes ways for students to pose problems and to search for and pursue different ways to represent, explore, and solve mathematical problems. In this process, students formulate conjectures and look for different ways to present and support mathematical results. Thus, problem solving is a way of thinking for students to understand mathematical concepts and to solve problems in which they engage in mathematical discussions with peers and teachers to make sense of ideas, to look for different ways to solve problems and to support solutions (Ding et al., 2022).

To characterize problem solving approaches, Polya (1945) focused on problem solving episodes that consistently appear during the solution process: **Understanding** and **making sense** of problem statements demand that students exhibit an inquiry approach to identify relevant data, to analyze the pertinence and consistency of the given information, and to delve into the concepts involved in statements and to clarify what is required to achieve or solve in the task.

Based on the understanding of the problem, the next phases involve the **design a solution plan** and ways **to implement** it to solve the problem, in this episode students rely on heuristic strategies to represent, explore, and work on problem solutions. Then, the **looking back** episode implies for students to check and review the process of getting or solving the problem and ways to generalize and extend the solution methods and to pose or formulate new or related problems. Polya's problem solving model has been seminal to frame research programs and teaching practices (NCTM, 2000; Schoenfeld, 1992; Toh & Tay, in press).

Schoenfeld (1985) proposed and implement a research program, that was inspired by Polya's work, with the aim to investigate the extent to which students' consistent use of heuristic strategies in approaching mathematical problems could help them become successful problem solvers. As a result, Schoenfeld posited that there are four interrelated dimensions to explain and characterize how students behave when they solve mathematical problems:

(1) **Knowledge base** or basic facts and resources that students bring to bear to work on problems. It includes facts, definitions, notation, procedures, algorithms to carry out mathematical operations, and ways to access and use those resources during the solution process.

(2) **Heuristic or cognitive strategies** that students rely on to represent, explore, and work on mathematical problems. Examining particular or simpler cases, working backwards, thinking of an analogous or similar problem, drawing figures, and using a table are examples of heuristics that can help students to make progress or overcome difficulties in their attempts to solve problems.

(3) **Control or metacognitive strategies** refer to ways for students to assess, monitor and control their own problem solving process. Monitoring and controlling the allocation of resources, including ways to change strategies or to adjust initial plans, are essential strategies for students to work on mathematical problems; and

(4) **Belief systems** that consider what students think of mathematics, about solving problem, and about themselves shape and determine the way in which they approach mathematical problems. Students developed their beliefs from their interaction with mathematical objects and from their accumulated experiences in their classroom work. Thus, the types of problems presented by teachers, the ways to assess the students' learning and problem solving performances, the textbook used, the homework, etc., contribute to the students' development of their beliefs or conceptions of the discipline. An example of one such belief is "Mathematics problems have one and only one right answer, there is only one correct way to solve any mathematical problem–usually the rule the teacher has most recently demonstrated to the class, …" (Schoenfeld, 1992 , p. 359).

Schoenfeld's framework has been widely used in mathematics education to support curriculum proposals and to structure problem solving learning environments. Furthermore, Schoenfeld also found that the heuristic strategies that Polya (1945) presented as relevant for problem solvers to work on mathematical problems were too general and complex to be used efficiently in problem solving approaches. Each strategy includes a set of sub-strategies whose application depends on particular features of the domain. For instance, to rely on the strategy "examining special cases" in a problem that involves roots of polynomials, a plausible candidate for a special case could be considering polynomials which are easy to factorise; but the same strategy for a problem with triangles, a special case could be examining equilateral triangles. Further, Schoenfeld (2023) suggested that for students to efficiently use heuristics they need to develop experiences in which they learn to use sub-strategies associated with each heuristic in different areas or mathematical domains.

Santos-Trigo et al. (2022) recognized that, in general, conceptual frameworks that characterize and explain problem solvers' behaviours came from documenting and analysing experts and students' problem solving performances that privilege the use of paper and pencil approaches to work and solve problems. Recently, during the COVID social confinement, teachers and students not only used different apps to present and discuss mathematical tasks, but students also had an opportunity to rely on "mathematics action technologies" to represent, explore, and solve mathematical problems. "Mathematical action technologies are those that can perform mathematical tasks and/or respond to the user's actions in mathematically defined ways" (Dick & Hollebrands, 2011, p. xii).

Some students further consulted online platforms to revise or extend their understanding of concepts and problem solving approaches. Hence, it is important that conceptual frameworks get updated to explicitly address not only the students' appropriation process to transform digital apps to understand concepts and to solve mathematical problems; but also, the type of reasoning they develop through the systematic use of digital tools in approaching mathematical tasks.

Santos-Trigo (2019) argued that the Dynamic Geometry System (DGS) such as GeoGebra represents a milestone development that provides a set of affordances for students to work on mathematical tasks. Specifically, students' construction of dynamic models of concepts and problems offers them an opportunity to engage in mathematical explorations that lead them to pose questions and to identify and formulate conjectures that later can be supported with mathematical arguments. Four categories or types of activities that students can work with the use of DGS are:

(1) Focusing on the **reconstruction of figures** that are embedded in problem statements. That is, the idea is that learners construct dynamic models of figures that often are included in problem statements. To this end, they think of the figure in terms of mathematical properties that are relevant to construct such models and, in this process, students identify new problems and relationships that are used to solve the task. For instance, if a figure involves a circle and an inscribed equilateral triangle, then students might decide between first drawing a circle and then inscribing the equilateral triangle or first drawing an equilateral triangle and then drawing the circle that inscribes such a triangle. Each path demands that the students rely on different concepts and strategies to identify mathematical relations that are important to solve the problem. As a result, students have different opportunities to analyse and contrast what concepts and resources are relevant in the reconstruction of the figure.

(2) **Investigation tasks** in which mathematical tasks become a departure point for students to engage in mathematical practices. That is, students transform, even routine textbook problems into a series of investigation tasks by changing initial conditions or extending the domain of the problem. In this process, students not only propose new problems, but also, rely on different concepts and strategies to dynamically model and explore problem extensions and solutions. Furthermore, students have an opportunity to change the initial nature of the problem into a set of nonroutine tasks that emerge from questioning initial conditions and intentionally looking for general cases that include extensions and generalization of solutions.

(3) Approaching **variational phenomena** in which students deal with optimization calculus problems via the construction of dynamic models without representing the problem algebraically. That is, students can generate a graphic representation of object's attributes that change in terms of tracing the loci of parameters associated with those changes. And based on this geometric approach students can parametrize the problem and construct the corresponding algebraic model that can be explored in terms of calculus concepts.

(4) The construction of **dynamic configurations** in which students are encouraged to ensemble or put together simple mathematical objects such as points, lines segments, triangles, rectangles, etc. and move some of them to find mathematical relationships and explore and support them through mathematical arguments. For instance, drawing a segment AB with A on the y-axis and a perpendicular line to the y-axis at A and the perpendicular bisector of segment A, point P is the intersection of the perpendicular to the y-axis and the perpendicular bisector of AB, then the locus of point P when point A is moved along the y-axis generates a conic section (parabola) (Figure 1).

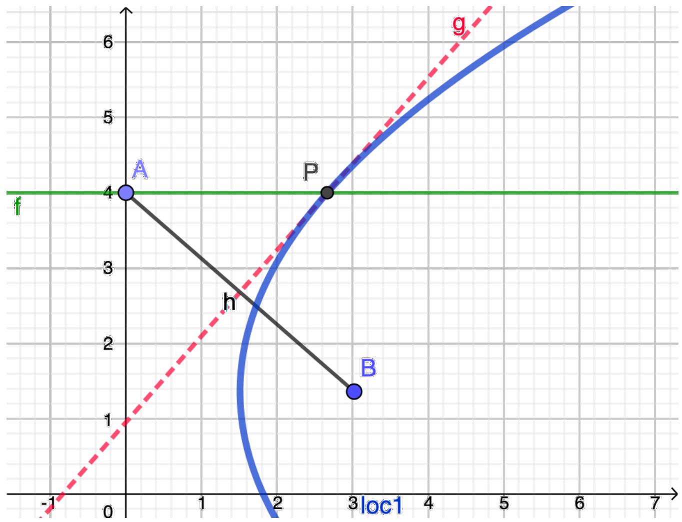

Figure 1. Construction of a simple dynamic configuration to generate a conic section

Santos-Trigo et al. (2022) proposed a digital tool for students to organize and monitor their problem solving approaches in terms of three interwoven categories: An inquiry or question method to delve into concepts and problem solving process; the coordinated use of both conveyance and mathematical action technologies (Dick & Hollebrands, 2011) to work on and discuss mathematical tasks; and a support system to foster synchronous and asynchronous discussions with peers and teachers.

A Mathematical Problem Solving Approach and the Use of Digital Technologies

In daily life, people make plans to move or get from one place to another, to go to the grocery store or to design an exercise routine. In this process, they pose questions and rely on units to measure parameters such as time, money, distance, speed, etc. to explore solutions paths and answers. Similarly, understanding and solving disciplinary problems requires that people delve into concepts, represent, and explore problem situations, to look for different ways to solve the problems, and communicate and support results or solutions. Romberg and Kaput (1999) suggested that curriculum activities that reflect mathematical practices:

> "… are those that involve students in problem solving and that encourage mathematization. Such tasks include situations that are subject to measure and quantification, that embody quantifiable change and variation, that involve specifiable uncertainty, that involve our place in space and spatial features of the world we inhabit and construct, and that involve symbolic algorithms and more abstract structures. In addition, they encourage the use of mathematical languages for expressing, communicating, reasoning, computing, abstracting, generalizing, and formalizing (p. 6).

Indeed, there are a variety of tasks in which problematic situations might emerge. Some examples of such tasks are:

1. How much active ingredient of an antibiotic will remain in a patient' organism who suffers an infection and takes one tablet of 16 units of active ingredient every 4 hours for 10 days and if it is known that the patient eliminates 50 % of the drug every 4 hours?
2. How many tennis matches will be played in a tournament to determine the winner if there are 1024 participants with a single elimination system?
3. Why do some tractor-trailers get stuck under certain underpass when the height of the bridge is clearly indicated?
4. How many apple trees could be planted in a land of three hundred square meters to maximize their production?
5. Can you find the area of a triangle whose vertices are its orthocenter, circumcenter, and the centroid?

The goal of the above tasks is for students to develop concepts, resources, strategies, and ways of thinking that lead them to formulate and solve a variety of problems situated in different contexts. Some of these tasks are discussed next to illustrate the importance of always looking for different paths to represent and solve mathematical problems, and how the use of digital technologies offers affordances for students to extend their ways to work and reason about problem solving approaches.

Kilpatrick et al. (2001) identified five strands for students to develop mathematical proficiency: **Conceptual understanding** that involves students' comprehension of concepts that includes addressing their meaning and ways to operate them, their relations and applications; **procedural fluency** that refers to the identification of operations and procedures and way to apply them appropriately; **strategic competence** that deals with ways for students to formulate, represent, explore, and solve mathematical problems; **adaptive reasoning** in which students have an opportunity of presenting, explaining, supporting, and reflect on mathematical concepts and problem solutions; and **productive disposition** to foster students' interest, motivation, and disposition to work mathematical tasks. How could these mathematical proficiency strands be integrated in a problem solving approach for students to learn the discipline? To address this question, a couple of examples are used to illustrate ways in which the students' use of digital technologies could provide affordances for them to engage in mathematical discussions.

In terms of students' productive disposition, the type of task and its context influence the students' interest and inclination to engage in mathematical thinking. Students might be interested in explaining how the process of taking certain medication works to treat a patient's infection and how the use of mathematics becomes important to understand the effect of the medication (task 1). In this case, students might be asked to consult information regarding active ingredients in medicine, dosage form, kidneys' filtering drugs, doses time, etc. Further, this type of task represents a family of phenomena that involves the variation of certain parameters that students can model with the use of mathematical concepts. The idea is that students problematize these phenomena and eventually pose a task that can be analyzed in terms of mathematical concepts and problem solving strategies.

Students consult information to both contextualize the phenomenon and to problematize it (Santos-Trigo, 2020). Using Polya's language, it becomes essential for students to make sense of the problem statement and to identify relevant information and ways to represent it. The use of a table provides the affordance for students to organize the information and to observe how the amount of active ingredient the patient accumulates during the treatment.

The questions that could be generated to analyze the problem statement of task 1 might include: What data are important to represent the problem? How can we relate the dose number, the time to take the medication, and the amount of active ingredient in the patient's system? How does the amount of the active ingredient behave during the patient's treatment? How can the relevant information be represented? What operations are involved in determining the amount of drug that the patient has in his system after receiving each dose?

Table 1 shows the amount of medication (active ingredient) that remains in the patient' system immediately after he took his tablet every 4 hours, knowing that his kidneys filter and eliminate 50 % of the ingredient during the four hours period. To determine the amount of ingredient for the second, third, etc. dose, the previous amount of ingredient is divided by half and then added to 16.

Table 1.
Doses and amount of active ingredient

Dose number	Elapsed time (every four hours)	Amount of the active ingredient after the corresponding dose
1	0	16
2	4	24
3	8	28
4	12	30
5	16	31
6	20	31.5
7	24	31.75
8	28	31.875
9	32	31.9375
10	36	31.96875

Students should be prompted to analyze the behavior of the amount of the active ingredient that appears in the third column of Table 1, which shows that after the fifth dose the amount of the active ingredient seems to get steady. Students could be prompted to consider what it means that the amount of active ingredient gets steady in terms of the medical treatment.

Graphical representation

Figure 1 shows a graphical representation of amount of the remaining active ingredient in the patient's system during the ten doses. The x-axis represents the time elapsed to take each dose and the y-axis represents the amount of the active ingredient. For example, point J has the coordinates (36, 31.97) and this means that the accumulated amount of active ingredient in the patient's system is 31.97 units after 36 hours (10^{th} dose) since the patient begun the treatment.

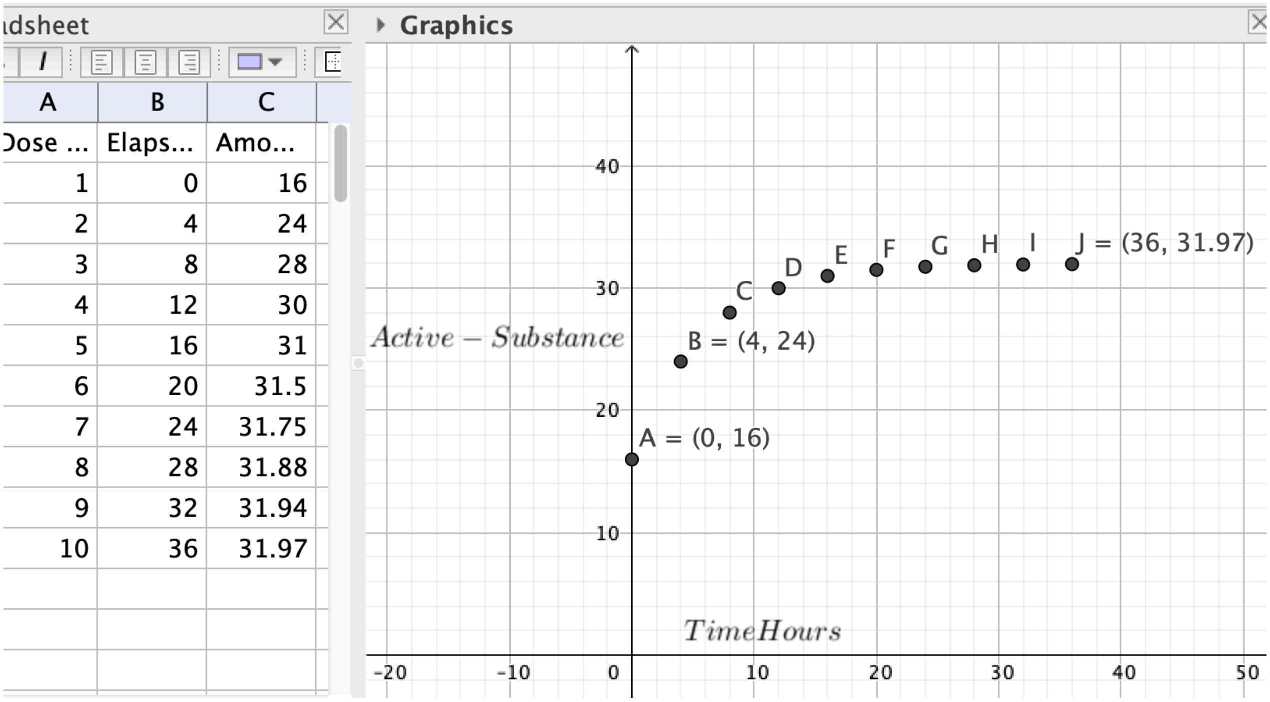

Figure 1: Amount of active ingredient after 10 doses

Looking for patterns

Is there any pattern that describes the way in which the patient accumulates the corresponding active ingredient after each dose? What operations are involved in calculating the amount of active ingredient associated with each dose?

Table 2 shows the procedures involved in determining the amount of ingredient that remains in the patient's system for each dose just before taking the next dose. It could be inferred that the use of fractions instead of carrying out the corresponding operation becomes important to identify the existing pattern. The first column shows the dose number and the second one includes the arithmetic operations to find the corresponding amount.

It is observed that the numerator in the expression that indicates the amount of ingredient can be expressed as: $16+2\cdot 16+2^2\cdot 16+\cdots+2^k\cdot 16$, where k is a positive integer. That is, if C_n represents the amount of active ingredient that remains in the patient's system in the n-dose then: 16 is a factor of C_n, the numerator includes powers of 2, with exponents varying from 0 to $n-2$, and the denominator can be written as 2^{n-1}. Based on this information, we have that:

$$C_n = \frac{16+2\cdot 16+2^2\cdot 16+\cdots+2^{n-2}\cdot 16}{2^{n-1}}$$

$$=16\left(\frac{1+2+2^2+\cdots+2^{n-2}}{2^{n-1}}\right)$$

Table 2.
Looking for a pattern in determining the amount of active ingredient

Dose number	Representing and finding the amount of active ingredient for each corresponding dose
1	0
2	$\frac{0+16}{2}=\frac{16}{2}$
3	$\frac{\frac{16}{2}+16}{2}=\frac{16+2\cdot 16}{2^2}$
4	$\frac{\frac{16+2\cdot 16}{2^2}+16}{2}=\frac{16+2\cdot 16+2^2\cdot 16}{2^3}$
5	$\frac{\frac{16+2\cdot 16+2^2\cdot 16}{2^3}+16}{2}=\frac{16+2\cdot 16+2^2\cdot 16+2^3\cdot 16}{2^4}$
6	$\frac{\frac{16+2\cdot 16+2^2\cdot 16+2^3\cdot 16}{2^4}+16}{2}=\frac{16+2\cdot 16+2^2\cdot 16+2^3\cdot 16+2^4\cdot 16}{2^5}$

Describing the pattern on the number of doses is transformed to finding a closed expression or formula. How can the sum $1+2+2^2+\cdots+2^{n-2}$ be represented? The problem initial conditions indicate that $2 \leq n$ since for n = 1, the amount of ingredient is directly determined for the table ingredient. If the sum is denotated as then, we have that:

$$\begin{aligned} S_{n-1} &= 1+2+2^2+\cdots+2^{n-2}+2^{n-1} \\ &= S_{n-2}+2^{n-1} \end{aligned}$$

And, we have that:

$$\begin{aligned} S_{n-1} &= 1+2\left(1+2+2^2+\cdots+2^{n-2}\right) \\ &= 1+2S_{n-2}. \end{aligned}$$

Then,

$$S_{n-2}=2^{n-1}-1,$$

that is, C_n can be expressed as:

$$C_n = 16\left(\frac{1+2+2^2+\cdots+2^{n-2}}{2^{n-1}}\right) = \frac{16}{2^{n-1}}\left(2^{n-1}-1\right) = 16\left(1-\frac{1}{2^{n-1}}\right)$$

Here, it is observed that for each amount remaining after every 4 hours another 16 units are added, then the accumulated amount of active ingredient in the patient's system will be:

$$Q_n = C_n + 16 = 16\left(1-\frac{1}{2^{n-1}}\right) + 16 = 16\left(2-\frac{1}{2^{n-1}}\right)$$

Figure 2 shows that algebraic model and its graphical representation of the amount of active ingredient that remains in the patient's body after taking each dose. It is observed that after a certain number of doses, the amount of medication (active ingredient) approaches constant. This means that for an interval of time, there is a constant amount of medication to fight the patient's infection. This explains why medical doctors will advise their patients to take certain medications for specific time to guarantee that certain amount of active ingredient will remain in the patient's system to fight an infection or illness.

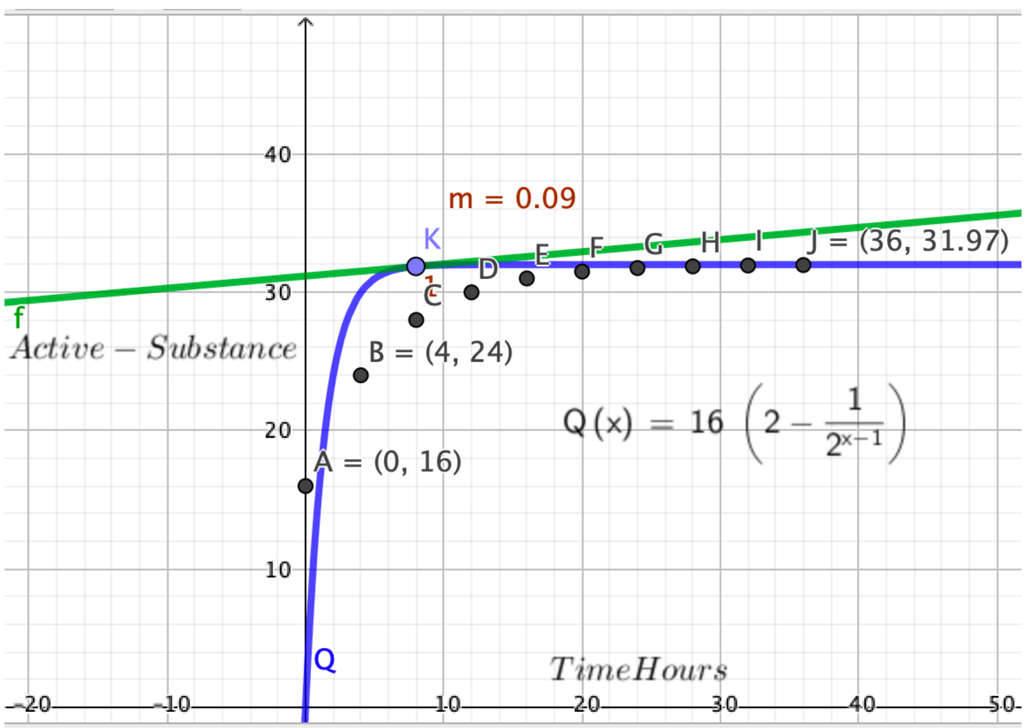

Figure 2. Graphic that represents the behavior of the amount of the active ingredient in the patient' system

It is also observed that the domain of the problem involves a list of discrete points associated with time intervals (every four hours); but the algebraic model (Q(x)) is continuous. The latter model is relevant for analysing the variation of the amount of the active ingredient in the patient's system.

In Figure 2, point K is any point on Q(x) and line *f* is tangent to Q(x) at K, the slope of line *f* varies when point K is moved along Q and the slope values becomes constant after certain time. In terms of the treatment, the stability of the slope means that during this period the medication will be effective to attack the infection.

In the five tasks that appear in the above list, the initial conditions were provided. Some conditions include specific data (medication, tennis, and apple tasks); but others demand that the students look for information to understand and make sense of the statement (tractor-trailer task). Thus, students have an opportunity to observe and problematize diverse situations that lead them to the formulation of problems. To reduce complexity associated with a situation, they need to assume certain conditions to represent and explore the phenomenon. For example, the constant active ingredient reduction or filtration of the patient's kidneys during the treatment is a relevant condition to represent and explore the amount of active ingredient that remains in the patient' system. The use of the tool, in this case GeoGebra, provided affordances to examine the task in terms of a table, a graph, and an algebraic model (Santos-Trigo et al., 2021).

Task 3, the tractor-trailers task, involves analyzing what parameters determine that trailers get stuck. Figure 3 is a two-dimension dynamic model that shows a representation of main trailer components such as its box, wheels, and road slope through corresponding geometric figures (rectangle, circles, lines, angle). Point M moves along the road and point S has the same *x*-coordinate as M. Point S also has the same *y*-coordinate the height of trailer measured from point P (base of the bridge) to point K. The locus of point S determines the path left by the length of the trailer' height which reaches its maximum value when passing through the underpass. Then, to clear the bridge, the height of the bridge needs to be higher than the maximum height that the trailer registers through the underpass.

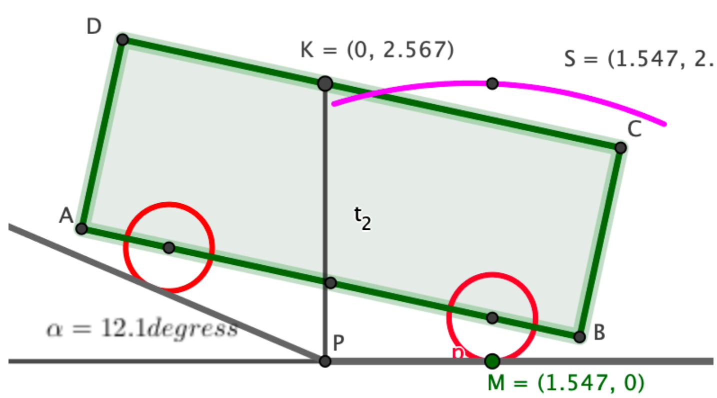

Figure 3. Dynamic model of the trailer passing that goes through an underpass

In this task, students have an opportunity to examine a realistic situation in which they need to identify what elements are relevant to explain the phenomenon. The use of GeoGebra provides the affordance to construct a dynamic model of the problem and to explore graphically how segment PK that represents the height of the trailer at the base of the bridge behaves. The next stage involves finding a curve parametrization that fits the locus of point S and then students can rely on calculus concepts (derivative) to find its maximum value.

To approach task 2, students could be prompted to pose questions to make sense of the statement and to look for ways to approach it. Is there any relation between the number of players and the number of matches to be played? What does it mean that a tournament follows a single elimination rule? What about if there are only four players? What properties does 1024 hold? What about factoring the number? What properties does this number hold? These types of questions might lead students to think of different strategies to determine the number of matches.

The solution approaches of task 2 could lead to the following problems:

1. A **direct** method. How many matches will be played in the first round? What about the second, third, or subsequent rounds?

$$512+256+128+64+32+16+8+4+2+1=1023$$

Is there another way to sum up this series of numbers?

$$1+r+r^2+\cdots+r^n=\frac{r^{n+1}-1}{r-1},$$

What conditions are needed to apply this formula? This expression can be applied for any natural n and any real $r \neq 1$. In this case, $r = 2$ and $n = 9$.

$$512+256+\cdots+2+1=2^9+2^8+\cdots+2+1=2^{10}-1=1023$$

2. A **labelling** approach. Another way of counting the number of matches is to assign a number to each player. That is, 1, 2, 3… 1024. Then, the first match is between players 1 and 2; the second match pits the winner of the first match against player 3, and so on. Thus, by following this matching system, there will be 1023 games to obtain the champion.

3. **Focusing on special cases**. Another way to solve this problem is to form groups of players to get each group winner. For example, we can think of 32 groups of 32 players in each group. Then, for each group there will be one winner, following the simple elimination rule. Thus, there will be 32 winners and the single rule elimination will now be applied to these 32 winners, etc.

4. **Focusing on the number of losers**. There are 1024 players who were enrolled in the tournament. If there is one winner, then there will be 1023 losers. Then, 1023 matches are needed to determine the winner since each loser loses only one match.

In task 2, students could be led to recognize that mathematical resources are important to make sense of the problem statement and to think of different ways to approach the task. Each approach demands that students identify different concepts and strategies to represent and solve the task. The first approach involves the sum of powers of 2 that relates to finding the sum of a geometric series and they have an opportunity to apply it. The second and the third approach involve the application of grouping and the use of special cases to represent the information and solve the task.

The fourth approach does not require number operations but relatse the number of games with the total number of losers.

Looking Back and Concluding Remarks

Mathematics curriculum proposals and teaching practices worldwide recognize that problem solving activities are essential for students to learn and construct mathematical knowledge (Törner et al., 2007). Indeed, both problem formulation and looking for different ways to solve problems are essential activities associated with the development of the discipline. In this process, the use of concrete tools (compass or rulers), semiotic (cartesian system), and digital technologies plays an important role in the process of posing and solving mathematical tasks (Santos-Trigo, 2022). It is evident that the use of digital technologies not only extends the activities and the structure of learning environments, but also, changes the content and organization of curriculum proposals (English, 2023).

The use of tools shapes the way people reason and work on mathematical problems. For example, the Babylonian civilization relied on concrete objects such as sticks and tables to develop and register arithmetic and geometry relations that were supported empirically. Later, the use of compass and straightedge became important for the Greeks to work on geometric problems and to develop the axiomatic method to present and support mathematical results. The Cartesian coordinate system, a semiotic tool, developed by Descartes in 1637 provides means and affordances to represent and explore geometry and variational problems (calculus) algebraically.

Today, the coordinated use of digital tools to represent, explore, and work on mathematical tasks not only opens new routes for students to approach those problems, but also provides affordances to extend ways of reasoning to solve the tasks (Santos-Trigo et al., 2021). For instance, students' construction of dynamic model of concepts and tasks requires that they think of the problem in terms of mathematical properties of objects used to model the problem. A set of heuristics such as dragging orderly objects within the model, tracing loci of some elements of the model, using sliders to explore variation of some object attributes, or measuring angles or segments become crucial strategies for students to identify and support mathematical relations that are relevant to solve and extend the initial problem.

Throughout the construction and exploration of the task model, students engage in problem posing activities and ways to extend the domain of initial tasks (Mason, 2000). For instance, the model associated with medical treatment can be applied to include other units of active ingredients, different numbers of hours to receive the drug, different rate of kidney filtration, etc. and these changes do not alter the deep structure of the model. The dynamic model can be applied to a family of cases that shares the same deep structure. Similarly, the tractor-trailer's model could be adjusted to explore cases that include variations in dimensions of the rectangle, radius of the circles, and road angle. This task is also an example of how a real situation can be problematized and approached in terms of mathematical concepts. Here, teachers play a fundamental role in orienting and guiding students to observe these types of phenomena and to think of mathematics as a way to represent, explore, and solve the problems that are associated with those phenomena.

Finally, conceptual frameworks to explain students' cognitive processes to work on mathematical tasks need to be updated to characterize the extent to which students transform digital tools in instruments to understand concepts and to solve mathematical problems. Likewise, it is important to analyse the reasoning that students construct through the consistent and systematic use of digital affordances and online developments to approach mathematical problems, share and discuss problem solutions.

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