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Metacognition in Mathematics Problem Solving: A Case of Making Monitoring and Regulation Visible

Shiqi JIN & LEONG Yew Hoong
National Institute of Education,
Nanyang Technological University, Singapore

This paper reports a two-student case study on how parts of 5th-grade students' metacognitive processes became visible while they solved a non-routine mathematical problem. The students used a designed Problem Solving Metacognition (PSM) worksheet that incorporated explicit metacognitive prompts for anticipated decision points, including understanding difficulty, inadequate progress, key insights, and uncertain answer. These prompts encouraged brief records of students' monitoring and regulation. A marker-response analysis traced how each student responded across the problem solving stages and revealed distinct metacognitive profiles. The case study suggests that the PSM worksheet may offer a practical way to make students' metacognitive actions more visible and analysable in regular mathematics problem solving classrooms.

Keywords: metacognition, problem solving, monitoring and regulation

Introduction

The development of students' problem solving capability has been considered a core aim in the curricula of many countries. Examples include the OECD's 21st Century Skills (OECD, 2018), the U.S. Common Core State Standards for Mathematics (NGA & CCSSO, 2010), the Pentagon Model in Singapore (Singapore MOE, 2021), and mathematical competencies proposed in China (China MOE, 2022).

Problem solving in mathematical classroom practice still faces challenges, making it difficult to be successfully implemented (Stacey, 2005). Nevertheless, there have been continuous efforts to promote and support the practical work of problem solving. The Mathematics Problem Solving for Everyone (MProSE) project in Singapore sought to infuse and diffuse problem solving in authentic mathematics classrooms (Leong et al., 2014). In China, large-scale evaluative examination reforms (such as the college entrance examination) have begun to reflect the emphasis on problem solving through the inclusion of non-routine tasks. On the students' part, empirical studies have revealed various obstacles faced by primary students in mathematical problem solving. Dong (2021) found that fifth-grade students' understanding of the problem tended to be superficial. They often read the problems quickly and moved straight to finding an answer, reflecting a fixed mindset and lack of ability to unpack implicit mathematical information. Zheng (2024) further pointed out that there were huge differences across ability levels: low-achieving students usually could not understand the problem context, middle-achieving students lacked persistence, and high-achieving students had poor systematic reflection habits, reflecting insufficient metacognitive ability.

The role of metacognition in supporting effective problem solving has been widely recognized (Lester, 1994; Schoenfeld, 1985; Veenman et al., 2006). Since "metacognition" was coined by Flavell (1976), the concept has been defined variously, but definitions seem to point to the cognizant thinking of one's own thinking and regulation of one's cognitive activities. In mathematics, three key components have become clearer: awareness, monitoring, and regulation. According to the Singapore Ministry of Education, the term metacognition refers to "thinking about thinking—the awareness of and the ability to control one's thinking processes, particularly the selection and use of problem solving strategies. It also includes monitoring and regulating one's own thinking and learning" (MOE, 2021, p. 14).

Researchers consistently argue that metacognition does not emerge spontaneously but requires explicit teaching and training (Zimmerman, 2008; Dignath & Büttner, 2008). Metacognitive self-questioning frameworks such as IMPROVE (Mevarech & Kramarski, 1997), the Problem Wheel (Lee, 2008), and STARtUP (Lee et al., 2014) have been reported to support problem solving and provide a means to elicit more productive teacher-student dialogue. IMPROVE operates across all stages whereas the Problem Wheel and STARtUP target the understanding-and-planning phase. However, both primary and secondary school teachers are often unprepared to offer instruction regarding metacognition. The instruction given tends to emphasize cognitive strategies while giving little attention to metacognitive ones (Dignath & Büttner, 2008). This indicates that teachers need further support in implementing explicit metacognitive guidance in the classroom (Kistner et al., 2010; Kramarski, 2018). The research by Schoenfeld (1985) argued that in problem solving, there are crucial points when students need to make important controlled decisions. Goos et al. (2000) showed the need for a problem solver to recognize warning signals and to trigger regulation to put the problem solving process back on track.

The study reported in this paper sought to render explicit metacognitive elements in students' problem solving. In order to make students' problem solving processes more transparent and informed, a designed Problem Solving Metacognition (PSM) Worksheet and an instructional intervention were used. These involved adding metacognitive checkpoints and regulative scaffolding to support students in tracing, monitoring, and regulating their thinking more efficiently, thereby improving their problem solving ability in general. Building on the Mathematics Problem Solving for Everyone (MProSE) project (Leong et al., 2014), the present study extends the metacognitive parts to address a research question related to students' problem solving processes: how do the students make visible their metacognition in their problem solving process as they work on the PSM Worksheet? In particular, this is a case study of how two students made visible their metacognitive responses during problem solving.

Problem and Problem Solving

Schoenfeld (1992) described two types of problems: anything that needs to be done or a task; or a perplexing or difficult question or task. Under the first definition, problems include routine exercises, wherein problem solving primarily serves as a tool for delivering conventional mathematical concepts, theorems, and formulas. Under the second definition, the difficulty must not be overwhelming and be presented as a challenge that is accessible to the student. However, the definition of what constitutes a problem is also relative to the problem solver (Goos et al., 2000). A task that presents a genuine obstacle to one may not be a problem at all for another (Henderson & Pingry, 1953). Lester (2013) described problem solving as an activity that involves various cognitive processes. These processes require specific knowledge and skills and also are influenced by various non-cognitive factors.

In this study, a problem is distinguished from the routine exercise. It is defined as one that does not have an obvious solution path or a ready-made procedure for problem solvers to follow, but it remains solvable with resources accessible for solvers with adequate struggle. Problem solvers need to draw on their existing knowledge, strategies, and resources to explore and find a solution. Problem solving is understood as an activity that provides opportunities for students to explore problem contexts, deepen understanding, extend strategies and solution, and develop self-reflection, rather than merely to obtain a correct answer. Problem solving processes include cognitive activities and the participation of non-cognitive factors. These factors enable students to monitor and regulate their thinking process and manage their cognitive resources. Overall, problem solving is regarded as both an important goal of mathematics learning and a major way through which mathematics is learned (Stein et al., 2003).

Mathematical problem solving is largely built upon Pólya's work. He conceptualized mathematics as problem solving and placed it at the centre of mathematics teaching. In his work *How to Solve It* (1945), Pólya proposed a four-stage framework for problem solving: (1) Understanding the Problem, (2) Devising a Plan, (3) Carrying Out the Plan, and (4) Looking Back. These four stages are well-known and requires no further elaboration. But it is important to note that these stages are not strictly linear but operate in a recursive and bidirectional manner. Pólya's work sparked extensive re-examination of problem solving instruction. At that time, the term metacognition however had not yet been coined. Metacognition in his work appeared only implicitly—for example, in the "looking back" stage. And subsequent research centered on heuristic instruction (Lester, 1983). Hatfield (1978) noted that numerous studies have employed various approaches targeting specific tasks with general heuristics to enhance students' problem solving abilities. But there is no reliable evidence that students' problem solving abilities were significantly improved through these approaches (Lester & Kehle, 2003; Schoenfeld, 2013). Lester (1983) and Schoenfeld (1983) argued that most efforts to enhance students' problem solving abilities fail largely because instruction overemphasized the development of heuristic skills while neglecting the management skills essential for regulating problem solving activities.

On the other hand, the MProSE project in Singapore built on Pólya's four-stage framework as a workable structure for classroom problem solving practice. In line with Pólya's emphasis on "looking back," the practical worksheets designed and used in MProSE included a "Check and Expand" section, where students were prompted to reflect on the work done earlier. However, its approach did not focus explicitly on metacognitive monitoring in practice. There were no explicit metacognitive prompts for students while solving the main problem itself.

Metacognition in Mathematical Problem Solving

Metacognition was recognized as a profoundly important factor intimately connected to problem solving (Lester, 1989). Metacognition according to Flavell (1976) is "one's knowledge about one's own cognitive processes or anything related to them" (p. 232). Building on this, Flavell et al. (2002) pointed out that metacognition consists of metacognitive knowledge, metacognitive monitoring, and self-regulation. In practice, however, it is hard to classify observable behaviors as specific monitoring or regulation. This difficulty arises because internal mental states are inferred from external behaviors. A single behavior may often reflect multiple elements of metacognition simultaneously. This classification ambiguity has been noted by several researchers (Nietfeld et al., 2005; Schraw & Moshman, 1995). It is widely accepted that metacognition can be categorized into two components: knowledge of cognition and regulation of cognition (Nietfeld et al., 2005; Schraw et al., 2006; Schraw & Moshman, 1995). The regulation of cognition is the conscious awareness of one's own cognitive processes, which often arises during planning, monitoring and evaluating of cognitive tasks (Schraw et al., 2006). In the domain of mathematics, Schoenfeld (1985) divided problem solving behaviors into two types: tactical and managerial. Tactical behaviors refer to the actions that are implemented, such as using algorithms and procedures. Managerial behaviors involve control processes that guide the problem solving activity. These behaviors include allocating resources, selecting appropriate strategies, and making decisions about how to proceed at critical points, determining when to abandon a particular approach, and evaluating what can be retained from unsuccessful attempts. These managerial behaviors are crucial for problem solving, and insufficient control can have a catastrophic impact on problem solving performance (Garofalo & Lester, 1985). The managerial and control behaviors discussed here correspond to the regulation of cognition in the above definition of metacognitive regulation of cognition. In this study, our investigation of metacognitive behaviors in problem solving is focused specifically on this category.

Schoenfeld (1985) highlighted turning points between episodes in problem solving, along with two additional managerial intervention points: (i) when new information or a new strategic suggestion arises, and (ii) when a sequence of strategic attempts fails, signaling the need for a strategic review. These managerial intervention points were referred to as the "make-or-break points" of problem solving. Goos et al. (2000) made a distinction between routine monitoring and more controlled monitoring. Routine monitoring involves regular evaluations at each stage to ensure the problem solving process stays on track, while control monitoring refers to explicit checkpoints where students are expected to detect difficulties and engage in corresponding regulatory processes as needed. Goos et al. (2000) identified three types of metacognitive monitoring "red flags": "lack of progress, error detection, and anomalous results" (p. 3). Developing problem solving ability requires students to experience "productive struggle"—meaningful effort in which students autonomously create at least part of the solution (Hiebert & Grouws, 2007). The value in productive struggle is not simply in coming up with the right solution. A set of experiments revealed that struggling with problems on one's own before, even to an incorrect solution, results in deeper learning and transfer than guided learning (Kapur, 2011, 2014; Schwartz et al., 2011). Thus, supporting students during this independent problem solving process in their productive struggle requires providing appropriate support. This support does not directly guide them to the answer but offers feedback related to their current actions, enabling them to adjust their behavior accordingly (Brousseau, 1997). In terms of scaffolding and metacognition, a detailed discussion was given by Holton and Clarke (2006). Comparing questions of scaffolding (Holton et al., 1997) and Wilson and Clarke's (2002) Metacognitive Action Cards, Holton and Clarke (2006) contended that scaffolding behavior corresponds to metacognitive behavior such that self-scaffolding literally corresponds to metacognition. In this research, we regard metacognitive scaffolding as intentional instructions with consequences of students internalizing them as self-scaffolding—that is, metacognition.

Based on the preceding literature review, it becomes clear that problem solving holds a core place in mathematics teaching and learning, and metacognition acts as a critical factor in enhancing productive problem solving. But how to productively implement problem solving and metacognitive instruction in everyday classrooms is not straightforward. In this context, we extended the Singapore MProSE project (Leong et al., 2016). By using Pólya-based Problem Solving (PS) worksheets and corresponding intervention lessons. The project integrates problem solving structures into mathematics lessons and provides teachers with a hands-on instrument to enact problem solving. But, again, as explained previously, MProSE only implicitly applied metacognitive aspects. From this knowledge, the current research incorporated explicit metacognitive scaffolding, which built on Goos et al.'s (2000) "metacognitive red flags" and Schoenfeld's "make-or-break points." This modification of the worksheet is to facilitate the investigation of the case of how students made visible their metacognitive activity while doing problem solving.

Methods

This section details the research methodology employed in this study. The research was conducted in three main phases: (1) Problem Solving Metacognition Worksheet design (2) mathematical problems selection; and (3) data collection, and analysis.

Problem Solving Metacognition (PSM) worksheet design

In this study, Pólya's classic four-stage problem solving framework was selected as the foundation for the worksheet design. This choice was informed by its clear and accessible structure, its alignment with experts' problem solving processes (Carlson & Bloom, 2005), and its continued use in practical contexts, including prior work such as Schoenfeld's undergraduate course (Schoenfeld, 1985), a mathematics problem solving course at the University of Georgia (Wilson et al., 1993), and the Mathematics Problem Solving for Everyone (MProSE) project in Singapore (Leong et al., 2016). The worksheet used in this study is a modified version of the problem solving worksheet developed in MProSE (see Figure 1**,** compressed by removing spaces for writing). Therefore, it was considered a suitable problem solving framework for primary students.

Based on the PS worksheet used in MProSE, this study adapted Pólya's original four-stage framework into three stages: Understand and Plan (UP), Do and Monitor (DM), and Check and Reflect (CR). Metacognitive elements were directly incorporated into the stage names, and the stage names were rephrased using language more accessible to primary school students. Pólya's Understanding the Problem and Devising a Plan were integrated to be the UP stage for the consideration of the cognitive load that a four-stage framework with corresponding metacognitive participation may impose on primary students. As noted earlier, primary students often tend to move directly to executing procedures. The new UP stage was intended to provide opportunities for students to engage in problem understanding and planning. The DM stage combined Carrying Out the Plan with concurrent monitoring and regulation, reflecting Schoenfeld's (1985) view that managerial control processes operate during problem solving execution. The CR stage corresponded to Pólya's Looking Back stage, with an explicit emphasis on reflection and evaluation. This three-stage structure aligns with research characterizing regulation of cognition as arising across planning, monitoring, and evaluating cognitive activities (Schraw et al., 2006). The underlying logic of the alteration was to convert the implicit metacognition into explicit and actionable metacognitive components in the worksheet, thus encouraging students to willingly engage their metacognitive participation during the process of problem solving.

Practical Worksheet

Problem

I Understand the problem

Use some heuristics such as Draw a Diagram, Restate the Problem, Use Suitable Numbers, etc. to help you.

I have understood the problem. (Circle your agreement below.)

Strongly Disagree Neutral Strongly Agree

1 2 3 4 5

You may proceed to the next page to work out a solution/partial solution.

II&III Devise a Plan and Carry it out

a) *State your plan clearly, for example: (i) Use Suitable Numbers and Look for Patterns; or (ii) Find the areas of all smaller triangles and work out their ratios.*
b) *Number each plan as Plan 1, Plan 2, etc.*
c) *Carry out the plan that you have stated.*

Plan 1 Statement of Plan:

Carry out Plan 1

IV Check and Expand

a) *Check your solution.*
b) *Write down a sketch of any alternative solution(s) that you can think of.*
c) *Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.*

Figure 1. PS worksheet used in MProSE

For the timing of metacognitive elements, this study adapted the concept of red flags proposed by Goos et al. (2000) which suggested that when students experienced “lack of progress, error detection, or anomalous results,” they needed more controlled monitoring and regulation; otherwise, problem solving was likely to fail. Lack of progress here was revised into the more neutral term “Inadequate Progress” to reduce students’ sense of frustration. “Error detection” was removed, since students often corrected themselves on the fly, making it difficult to capture as a stable decision point in a paper environment, while also avoiding overlap with answer-level points. “Anomalous results” was refined as “uncertain answer”, which better captured students’ doubtful psychological status in the checking stage. We also included “understanding difficulty”, because students were likely to find it hard to start solving the problem as they had problems in unpacking the task.

In addition, we also incorporated Schoenfeld’s (1985) make-or-break points, which suggested that problem solving involves both “break points,” where one must stop and adjust, and “make points,” where new information creates opportunities for strategic advancement. Here, Schoenfeld’s conception of “make points” generated by new information in group environments was translated to “key insights”, a construct better suited to individual written sheet solving of problems. When students spotted a new clue or pattern, they were asked to intentionally think about how to capitalize on it for decision-making. The PSM worksheet contained four metacognitive management decision points: understanding difficulty, inadequate progress, key insights, and uncertain answer. These four points served as the essential timing anchors for incorporating metacognitive prompts in the worksheet.

Goos et al. (2000) and Schoenfeld (1985) did not provide students with corresponding metacognitive regulatory supports. This study was designed not only to help students become aware of these decision points but also to provide appropriate metacognitive scaffolding to support effective decisions and regulatory control. For this purpose, as outlined in the literature review, metacognitive questioning was adopted as the scaffolding, and the specific design underwent multiple iterations.

Consequently, the final worksheet was redesigned into two sections: a Problem Solving Workspace and a Metacognitive Engagement section (see Figure 2, compressed by removing spaces for writing). The right-hand metacognitive engagement section functioned as a toolbox that was always available but not mandatory. Lengthy metacognitive questions were simplified into lower cognitive demand formats of check boxes and short self-report statements. This design transferred partial control of metacognitive scaffolding to the students, enabling them to access support actively and selectively, thereby maximizing the utility of prompts while minimizing their interference.

For the "Understanding Difficulty" point during the Understand and Plan stage, prompts were designed to help students start by clarifying the problem. A simple question like "What have you understood so far?" encouraged students to summarize their current understanding. Then they were prompted to reflect upon their first plan, helped by a list of potential heuristics related to the task. Additionally, students were encouraged to identify potential difficulties they should pay attention to. During the Do and Monitor stage, two points were addressed. (a) Key Insights: Students were first asked to briefly describe their important discovery. Then, through checkboxes, they were prompted to consider how this insight might create new opportunities for solving the problem. (b) Inadequate Progress: Checkboxes were provided to help students activate specific regulatory strategies they can use when they feel stuck. For the "Uncertain Answer" point in the Check and Reflect stage, the worksheet guided students to first write down their preliminary answer and briefly explain why it seems unreasonable or odd. Subsequently, they were presented with checkboxes to prompt corresponding corrective actions, encouraging them to engage in appropriate checking or revision behaviors.

The final PSM worksheet reported in this study represents a balanced outcome reached after multiple rounds of iteration and balance. We calibrated the degree and timing of metacognitive prompts to avoid either disrupting students' problem solving flow or providing insufficient support. The prompts were ultimately anchored to key metacognitive decision points (understanding difficulty, inadequate progress, key insights and uncertain answers). This balance ensures that the prompts function as scaffolds when necessary while minimizing interference with problem solving activity. Meanwhile, we balanced prompts that were too general with those that were too task-specific to address the variability of students' metacognitive needs. The PSM worksheet represents an initial theoretical attempt of problem solving to embed more explicit metacognitive elements into Pólya's framework.

Mathematical problems selection

The aim of task selection in this case study was to highlight the acts of monitoring and regulation processes during problem solving. Accordingly, non-routine problems were used, as such problems are typically not confined to a specific mathematical topic, lack fixed solution procedures, and require the use of one or more heuristic strategies, thereby placing demands on the coordination of multiple cognitive processes (Foong, 2009). These characteristics

Problem Solving Worksheet with Metacognition

Problem: **I. Understand and Plan (UP)** Use heuristic (backwards/try a smaller number/conjecture and verification/estimation, etc.) to help your understanding. ☐ I have understood the problem. (You may proceed to the next page to work out a solution/partial solution.) ☐ I don't know how to start (You can use the right column to help you understand)	**Understanding Difficulty** a)What have I read so far? (in one or two brief sentences) b) I initially going to solve this problem with ______________________ The difficult part is______________.
II . Do and Monitor (DM) a) Start your attempt and record your process clearly. b) Fill in the right column when the following happens: • You gain an important insight (Fill in A) • You experience a lack of progress (Fill in B) ☐ I'm on track **(You may move to next stage)**	**A. Key insights ☐** My important discovery (1–2 sentences, e.g. new clue/pattern/relationship/constraint): ___________________________ ___________________________ These helped me to (tick): ☐ Revise understanding ☐ Check my strategy ☐ Find a new path ☐ Gain confidence ☐ Other: _____________ **B. Stuck / No Progress☐** In response, I will (tick): ☐ Keep and refine ☐ Try another strategy ☐ Break into smaller steps ☐ Back to Understand ☐ Pause for now ☐ Other: _____________
III. Check and Reflect (CR) a) Check your solution (If Uncertain Answer appears, fill in section C on the right) b) Go beyond your solution c) Write down another possible method or idea,if any (sketch or keywords is OK). ☐ My answer makes sense. (Well done on completing the problem!)	**C. Uncertain answer ☐** My initial answer: ______________ Why I think it's unreasonable or uncertain: ____________________________________ I will (tick one or more): ☐ Check special values ☐ Work backward ☐ Trace my steps ☐ Compare with another method ☐ Accept for now ☐Other: ______

Figure 2. The final PSM worksheet

correspond directly to the study's definition of a mathematical problem as distinguished from a routine exercise. Specifically, problems were chosen based on the following criteria: (a) Exploratory demand—to reduce interference from students' prior knowledge and beliefs, a low-floor–moderate-ceiling principle was applied, such that tasks were easy to start while still leaving space for extension; as a result, students would likely feel motivated or compelled to search for a solution. (b) Productive struggle—problems lack ready-made procedures but remain solvable with resources accessible at this grade level with adequate struggle. (c) Metacognitive decision points—tasks were designed to elicit re-representation and initial strategy selection, to support sustained exploration with on-line monitoring and regulation, and to prompt plausibility checks and brief reflection.

Part of the problem set originated from the MProSE project and the difficulty level was modified by adjusting number ranges and step complexity, simplifying wording, and localising contexts. The "Nice Number problem" met these criteria and was therefore selected as the focal task for analysis, as it is easy to start, offers room for extension, and availability of potentially useful prompts for students to engage in monitoring and regulation throughout the problem solving process. The final task list is shown in Figure 3.

There is a need for a teaching intervention in this study, because students would otherwise not be familiar with the metacognitive prompts. As this intervention is not the main focus of this study, it will be explained briefly. The intervention consists of six lessons over six days, with each session lasting one hour. Each lesson was anchored to a specific instructional focus, and problems were purposefully chosen to elicit the corresponding metacognitive monitoring and regulation focus. The overall lesson unit is displayed in Table 1.

Data Collection and Analysis

Data collection was conducted through the Problem Solving Metacognition (PSM) Worksheet, which recorded students' real-time identification of metacognitive make-or-break points, strategy selections, and regulative actions, reflecting the on-line visibility of metacognitive processes during problem solving. The data were gathered in the final lesson of the intervention using the Nice Numbers task. The students had 60 minutes of independent work: no discussion and no teacher assistance. The specific task statement and possible student solutions are shown in the figures below (see Figure 4).

The case study was conducted with two fifth-grade students, Emma and Andy, to illustrate how students, within the PSM framework, made their metacognitive monitoring and regulation visible at different stages of problem solving. These two students were selected from the intervention group to form a case precisely because of the distinctiveness of their metacognitive pathways. Among the two students, one began the task without a complete plan or effective strategy. After encountering difficulties and engaging in productive struggle, she adjusted her approach through metacognitive regulation and shifted to a more effective strategy. The other student started with a more global plan and clear strategy, and through continuous local monitoring and regulation, gradually refined and optimized his solution.

These two students were drawn from a larger intervention study group of 14 volunteers (6 girls and 8 boys, aged 11-12) recruited from a fifth-grade class at a public primary school in Shenzhen, China. All participants were recruited through the researcher's personal connections with teachers and parental consent was obtained.

Problem1: Handshake problem

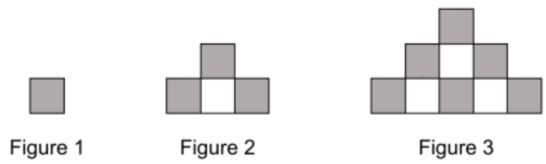
Twenty students from different schools joined a summer math camp. On the first day, they played a handshake game to get to know each other.

In this game, each student shakes hands with every other student once.

How many handshakes will happen in total?

Problem 2 : Square pattern

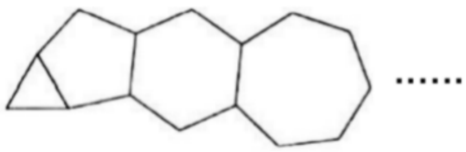
The diagram shows a sequence of patterns formed by identical squares.



A figure in the pattern has 60 more shaded squares than unshaded squares. What is the total number of squares in the figure?

Problem 3: Shape Puzzle

How many sides does the shape formed by placing together a triangle, a regular pentagon, a regular hexagon, a regular heptagon, ... , a regular 65-gon, all with side length 1?



Problem 4: Sum of All Digits

Add up all the digits of every integer from 1 to 99. What is the total sum?

Problem 5:New average

An integer is removed from a set of positive integers {1, 2, …, n}. The average of the remaining set of numbers is 19/4(or 4.75). What is the removed number?

Problem 6: Nice Number

Some numbers can be written as the sum of two or more consecutive positive integers. For example: 3 = 1 + 2; 9 = 4 + 5 or 2 + 3 + 4； We call these numbers “nice numbers.” Which numbers from 10 to 30 (inclusive) are nice numbers?

Figure 3. Mathematical problem set

Table 1.
The intervention lesson unit

Lesson	Problem	Objective	Activities
1	Sum of odd numbers (1+3+…+99); Handshake	Introduce common used heuristics & PS stages UP–DM–CR;	Students first solve odd-sum → teacher "expert" demo to introduce stages & heuristics → students independently solve Handshake → collect work
2	Squares Pattern (difference of 60)	UD (understanding difficulty monitoring and regulation)	Review L1 (student work, think-aloud) → explicit teaching of UD → students complete Squares Pattern on PSM
3	Shape Puzzle (outer boundary after tiling)	KI & IP (insights & progress monitoring and regulation)	Review L2 → explicit instruction of KI/IP → students solve Shape Puzzle independently on PSM
4	Sum of All Digits (1–99)	UA (plausibility checks & reflection)	Review L3 → explicit instruction of UA → students solve Sum of All Digits on PSM
5	New Average	Full-points integration in PSM worksheet (UD/KI/IP/UA)	Review L4 → complete New Average with full PSM flow; teacher scaffolds fade to encourage self-monitoring and regulation
6	Nice Numbers (10–30)	Independent use of stages & metacognitive points; self-report metacognition	Individual problem solving on PSM worksheet (no discussion) → collect scripts

The coding framework for analyzing students' metacognitive monitoring and regulation in the PSM worksheet was developed by integrating prior models from Goos et al. (2000) and extending them to capture both routine and more controlled metacognition. In addition to difficulty-related markers, the framework also includes Key Insights as opportunity markers—moments when students identified patterns or relations that enabled redirection or acceleration of progress. These insights are treated as markers rather than responses, since their origins cannot be precisely determined.

Initially, our coding attempted a strict one-to-one mapping between metacognitive makers and regulations according to PS stages (Figure 5). However, as the analysis progressed through multiple rounds of refinement, it became evident that many regulatory response behaviors were repeated across stages and triggers, and that responses typically operated at two distinct levels. This observation resonates with Garofalo & Lester's (1985) global and local metacognitive behaviors and Goos et al.'s (2000) differentiation between routine and controlled metacognitive monitoring and regulation. Our final framework therefore codes metacognition as markers and responses (Figure 6). Metacognitive markers were classified as red flags (Understanding difficulty, Inadequate progress, Uncertain answer), opportunity marker (key insights), and routine marker (Local monitoring). Metacognitive responses are further distinguished into local/routine and global/controlled levels. By capturing both immediate tactical adjustments and strategic shifts, this framework helps to interpret the responses on the PSM worksheet as a metacognitive trajectory. The trajectory will become clear when describing the problem solving processes of Emma and Andy.

Problem 6: Nice Number

Some numbers can be written as the sum of two or more consecutive positive integers.

For example: 3 = 1 + 2; 9 = 4 + 5 or 2 + 3 + 4; We call these numbers "nice numbers." Which numbers from 10 to 30 are nice numbers?

Solution 1: Enumerate 10–30

Work through each number from 10 to 30 and try to express it as a sum of two or more consecutive positive integers. Keep successful cases and cross out failures. This brute-force approach is accessible, but it is harder to be certain about 16: many tries fail, and confirming "not nice" can feel inconclusive with enumeration alone.

Solution 2 : Grow patterns from small cases

Start with small runs and extend.

2-strings: all odd numbers in 10–30 are nice (11, 13, …, 29).

3-strings: find 12 (=3+4+5), 18 (=5+6+7), 24 (=7+8+9), 30 (=9+10+11); remove duplicates 15, 21, 27

4-strings: add 10 (=1+2+3+4), 14 (=2+3+4+5), 22 (=4+5+6+7), 26 (=5+6+7+8); remove duplicates 18, 30

5-strings: add 20; remove duplicates15, 25, 30

6-strings:all duplicates (21,27)

7-strings: add 28 (=1+2+3+4+5+6+7).

8-strings: smallest is 36, which is beyond 30.

Collecting these shows that every number from 10 to 30 is nice number except 16.

Notes on task selection

Students must unpack a new definition ("nice number") without a ready-made formula, which naturally creates Understanding Difficulty. Fifth graders have limited algebra and no formal arithmetic-progression tools, so they must adjust when stuck (an IP moment) and notice patterns (a KI moment). Solution may initially be incomplete, prompting looping back in CR stage for further checking and supplementation. Even with the conclusion "only 16 is not nice," students naturally question whether they missed this case—inviting informal generalisation and verification. Given the grade level, a rigorous proof that 16 is not nice number is not expected.

Figure 4. The final task "nice number" and possible solution for primary students

Metacognitive make-and-break points	**Triggering Conditions**	**Regulatory Actions** **(Observable Behaviors)**
Understanding Difficulty	Unclear information Ambiguous problem statement Lack of prerequisite knowledge	Clarification Seeking(restate/clarify problem) Constraint Setting(Establishing boundaries or conditions, marking risky options,) Heuristic Activation (recall/use prior strategies or knowledge)
Inadequate Progress	Low efficiency Repetitive failures Deviation from goal Error detection	Strategy adjustment(shift or modify approach) Goal-directed filtering(eliminate irrelevant options) Trade-off decision(time, speed,space)
Key Insights	Recognition of a pattern, Discovery of structural features	Pattern Noticing (spotting regularities) Insight Elaboration (articulating/expanding on the insight) Incorporating Insight (integrating discovery into solution process)
Uncertain Answer	Doubt about correctness or validity of answer	Local revision(minor adjustments) Systematic correction (comprehensive reworking) Alternative attempt (trying a different method) Adaptive justification (defending or rationalizing an answer) Reflective abstraction(drawing general conclusions, abstracting principles)

Figure 5. The initial code framework for analyzing metacognitive behaviors

Metacognitive Markers	
Red flag	Understanding Difficulty (UD)
	Inadequate Progress (IP)
	Uncertain Answer (UA)
Opportunity marker	Key Insights (KI)
Routine marker	Local Monitoring (LM)

Metacognitive Responses (PSM Observable Behaviors)	
Local/ Routine responses	**Task clarification(TC):** Restate or represent the structural properties or constraints of the problem
	On-the-fly sub-goal checking(OC): Quick check that current step serves the present sub-goal
	Take stock(TS): Brief backward look to summary of what's been done so far
	Method flexibility (MF): Small method adaptations or combining within the same overall strategy.
	Error detection(ED): Spot and mark an local error
	Self-explanation(SE): Brief written notes or annotations explaining a step or recognizing a pattern.
	Trade-off decision(TD): Small adjustment due to time/space constraints
	Partial Correction (PC): Limited revision of selected parts without full systematic coverage.
Global / Controlled responses	**Strategy selection(SS):** Initial selection of a strategy
	Strategy transformation(ST): Change or restructure of strategy
	Systematic correction(SC): Deliberate, orderly re-check and repair of a set/segment

Figure 6. The final Metacognitive Markers-Responses framework

Findings

This section focuses on how Student Emma and Student Andy made their metacognition visible during their problem solving attempts.

Emma's metacognitive profile

Emma's problem solving process on the PSM worksheet revealed the triggering of multiple metacognitive markers and corresponding regulatory responses, forming a metacognitive "monitoring–regulation" trajectory.

<u>Understand and Plan stage</u>

She encountered an Understanding Difficulty (UD) metacognitive marker, which in turn triggered both task clarification (TC) and strategy selection (SS) metacognitive responses. Here, TC is considered a local response, while SS is coded as a global response. This is because, although for Emma the strategy selection was not a comprehensive or overarching plan, rather, it functioned as an initial step to get her started, the choice of strategy was still crucial for pushing the problem solving process forward. For understanding the problem, Emma restated the problem and highlighted key information to clarify it (see Figure 19 "Use two or more numbers to represent a number, find those numbers between 10 and 30 that can be represented by two or more such numbers and be careful that the numbers must be consecutive"). Through this task clarification she tried to unpack the definition of nice number and reminded herself of the conditions, such as within the range of 10-30. Then based on the problem, she activated her prior heuristic strategy knowledge to choose table listing as her initial strategy (see Figure 7: she explicitly recorded her intention to first use the table listing method to solve this problem). This series of metacognitive responses: task clarification (TC) and strategy selection (SS) enabled her to make explicit her metacognitive knowledge of the task and strategy.

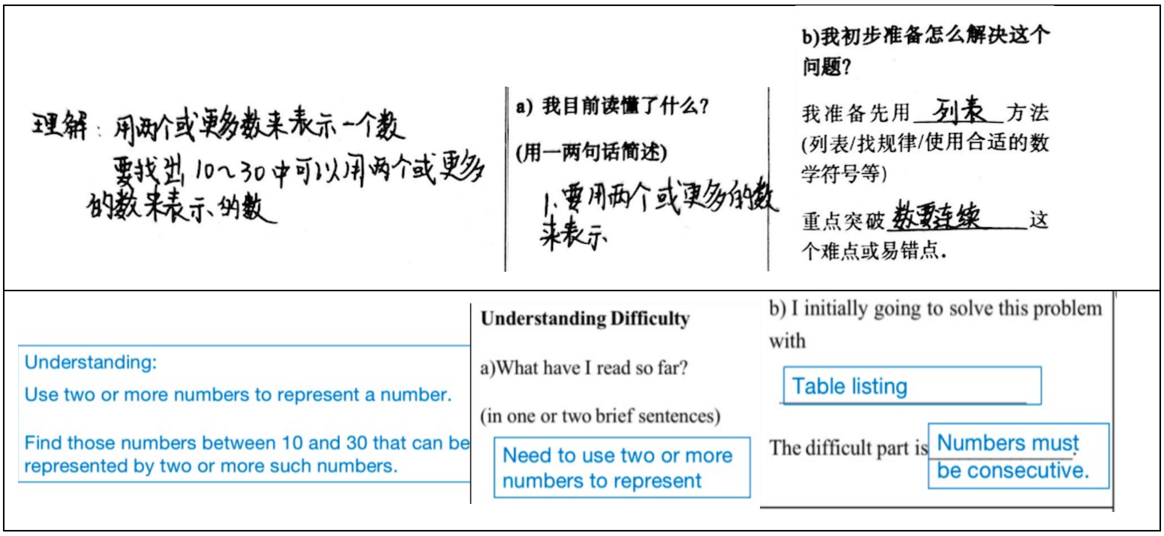


Figure 7. Emma's task clarification (TC) and strategy selection (SS) in UP stage

<u>Do and Monitor stage</u>

Four types of marker-response patterns were observed in Emma's PSM worksheet. (a) an inadequate progress (IP) marker combined with a key insight (KI) marker leading to a strategy transformation (ST) response; (b) instances of local monitoring (LM) markers gave rise to trade-off decisions (TD) responses; (c) a further key insight (KI) marker prompting method flexibility (MF) response; (d) local monitoring (LM) markers also elicited error detection (ED) responses.

Firstly, (a) Emma exemplified both inadequate progress (IP) marker and key insights (KI) marker, which together triggered a strategy transformation (ST) metacognitive response. On her worksheet she used a random trial-and-error method and recorded repeated unsuccessful attempts in expressing 10 and 11 as sums of consecutive integers (see Figure 8: 10 = 2 + 3 + 5 / 2 × 5, crossed out, not a "nice number", crossed out; 11 = 1 + 2 + 3 + 5, crossed out). The repetitive failures and inefficient method elicited her awareness of an inadequate progress (IP) metacognitive red flag. As she ticked 'try another method' under the corresponding inadequate progress in the metacognitive column of the worksheet (see Figure 8), which illustrates her metacognitive monitoring and regulation.

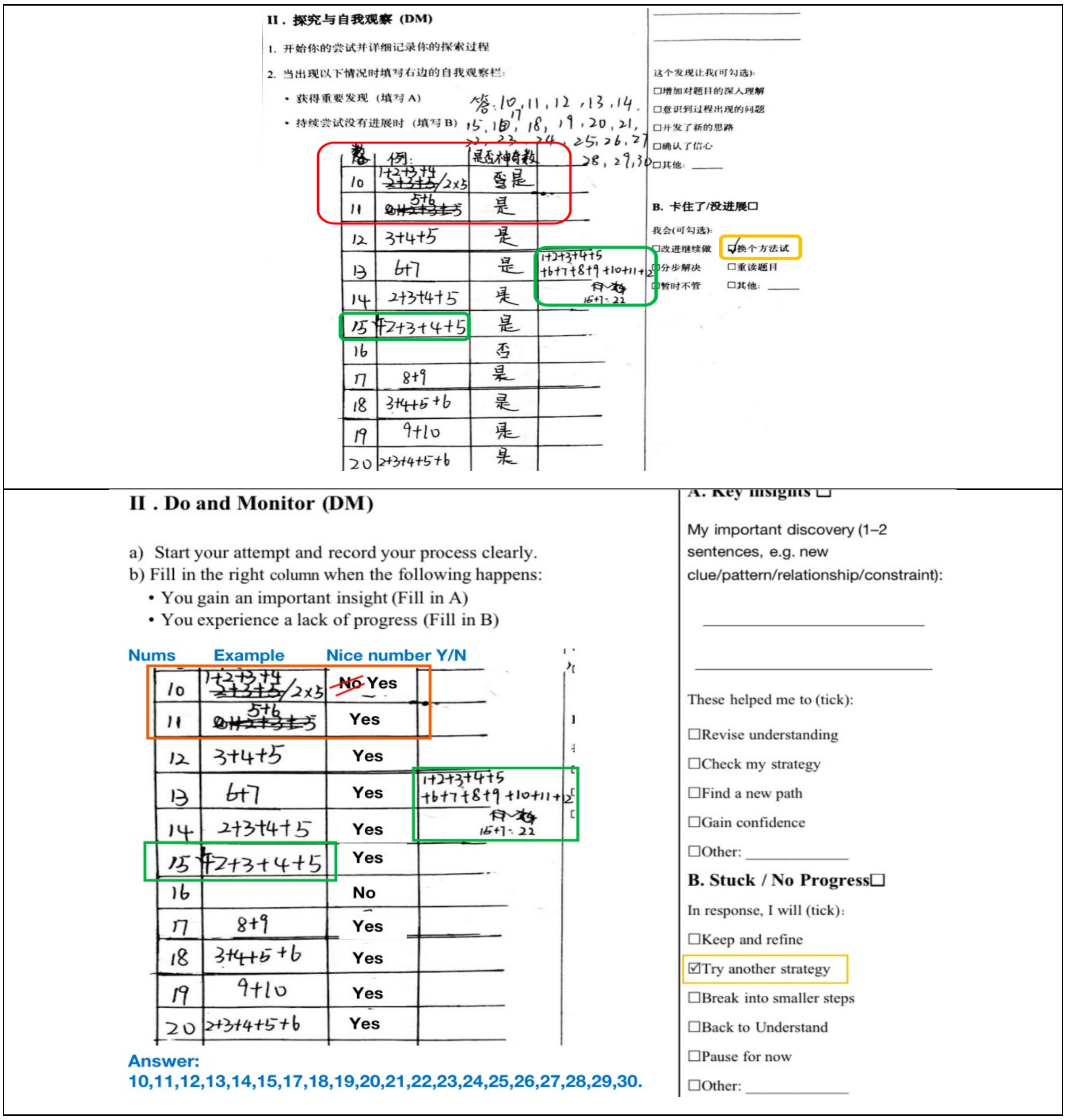


Figure 8. Emma's DM stage part A

Subsequently, Emma experienced a key insight (KI) metacognitive marker, when she attempted to represent 10–12 (10=2+3+4,11=5+6,12=3+4+5), she then saw them as sub-strings of the longer 1–12 string (see Figure 8. as noted in the remarks column for 13, in which Emma wrote down the long consecutive sum expression 1+2+3+…+12). For 13, she only extended the sum up to 12, it seems that this was intended to ensure that if 13 had a valid consecutive representation, it must lie within this range. This marks a strategic shift from random attempts to deliberate selection of sub-strings from the longer string**.** Following this, she was able to work out the representations for the numbers 13–20 more smoothly (as evidence: in comparison to the above, there were hardly any cancellations from 13–20; her representation of 15 is 1+ 2+3+4+5 rather than 7+8, and her decisive marking of 16 as "not nice") (see Figure 8). Although she did not write an explicit explanation why 16 is not a nice number, it is likely that the systematic sum of sub-string up to 12 helped her see that no combination within that range could produce 16.

(b) When Emma attempted to write 20 as a sum of consecutive positive integers, her work was marked by a local monitoring (LM) marker when the speed and inefficiency of her work became apparent in her written process. This awareness was followed by a trade-off decision (TD response): she abandoned the time-consuming tabular format and adopted a simpler line-by-line listing for numbers 21–30. Although there were slight adjustments in format, she still relied on the sub-strings of long expression from the previous page (see Figure 9 for her answer for 21–25, e.g. 21=1+2+3+4+5+6).

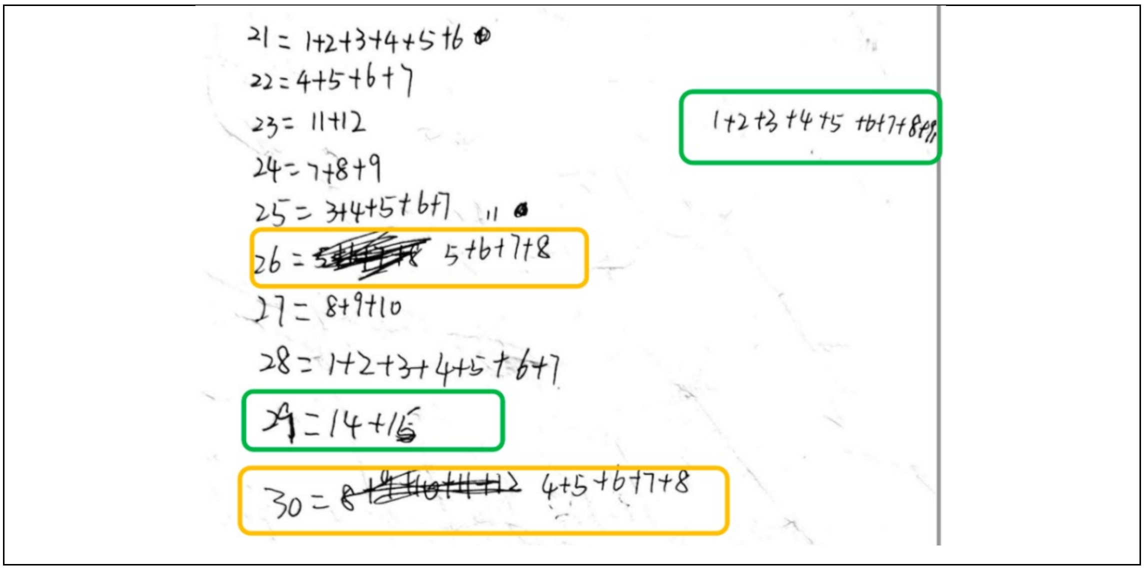


Figure 9. Emma's DM stage part B

At the row for 23, Emma noticed the inefficiency of repeatedly turning pages and signaled another local monitoring marker, which she addressed through a further trade-off decision (TD) response. She wrote "1+2+3+…+9" directly on the new page at the end of the row for 23 to streamline her procedure. Here she decided to stop at 9, rather than extend the strings to 12. This range may reflect a pragmatic judgment of sufficiency and suggests a possible reliance on earlier results in regulating her work.

(c) Emma's representation of 29 as a sum of consecutive integers marked a method flexibility (MF) response, following a key insight (KI) marker. It is likely that because a number of the odd numbers shown in the previous page (e.g., 13, 17, 19 …) can be written more simply as two consecutive integers, she would have identified an alternative strategy for odd numbers. As evidence, she wrote "14+15" for 29, even though this pair was not contained in the previously constructed long strings (1–12 or 1–9). This local method change from sub-string checking to direct writing of odd numbers as sums of consecutive integers can be interpreted as an instance of her flexible regulation across different methods.

(d) During exploration, Emma repeatedly displayed a local monitoring (LM) marker that elicited error detection (ED) response. Rather than checking until the end, her in-process error corrections at 10, 11, 26, and 30 (see Figure 8 and Figure 9) suggest that her local monitoring and error detection were embedded within the problem solving flow itself.

It is possible that her frequent local monitoring and in-process error detection reduced the extent to which further checking was recorded in the CR stage. Thus, her record in the worksheet does not show any systematic checking or overall justification.

Andy's metacognitive profile

Emma only had a systematic strategy emerging after many struggles (evidenced by the cancellations), whereas in Andy's case, there appears to be less struggle initially, but a clear strategy right from the start and he tweaked it along the way as he executed it.

Understand and Plan stage

Andy displayed a striking key insight (KI marker) which was followed by strategy selection (SS response). He noted that "all odd numbers" and "the sum of two consecutive integers cannot be even" (although his note omitted the word consecutive), and further added that "three consecutive integers yield an even number only when starting with an odd term, while four consecutive integers always sum to an even number" (see Figure 10). This early structural awareness enabled him to adopt a parity-based strategy from the outset: to first enumerate all odd nice numbers between 10 and 30, and then proceed systematically to explore even nice numbers through four-strings and three-strings. The KI enabled his early strategic orientation and provided a global plan for his subsequent work.

Andy's notes	Translation
所有的奇数 [illegible] 两个数加起来不可能等于偶数 三个数加起来只有奇数开头才有偶数 四个数加起来一定等于偶数	All odd numbers [illegible] Two numbers added together cannot be an even number. Three numbers added together give an even number only if the first one is odd. Four numbers added together will always give an even number.

Figure 10. Andy's notes in the UP stage

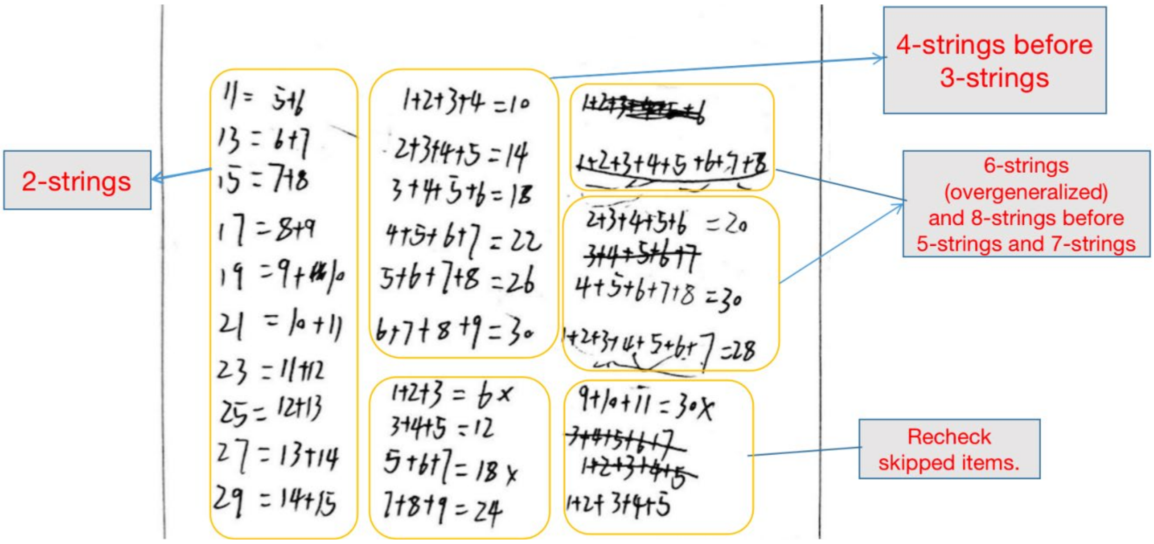


Figure 11. Andy's notes in the DM stage

Do and Monitor stage

Andy's local monitoring triggered a series of local responses: continuous on-the-fly sub-goal checking (OC), method flexibility (MF), and periodic stock tack (TS). His local monitoring marker to on-the-fly sub-goal checking response was evident in his consistent attention to keeping results within the 10–30 range. For instance (see Figure 11), he directly listed the odd numbers from 11–29 with their consecutive-sum representations (e.g., 11=5+6, 13=6+7, ⋯, 29=14+15). When exploring even numbers, Andy skipped the 3-strings before 4-strings because he thinks these would produce both odd and even numbers ("three consecutive integers yield an even number only when starting with an odd term"). He went first to 4-strings as he was sure they would definitely yield even numbers ("four consecutive integers always sum to an even number"), thereby was coded as a method flexibility (MF). Only when he noticed that there were still some even numbers did he move back to 3-strings. During the process, he repeatedly engaged in on-the-fly sub-goal checking (OC), crossing out out-of-range or redundant results while retaining those within the range. After completing the two, three, and four consecutive cases, he conducted a TS check, which is evident from his listing of the numbers. He recorded all confirmed nice numbers on a new page and separately listed 16, 20, and 28 with crosses (see Figure 12), provisionally marking them as non-nice.

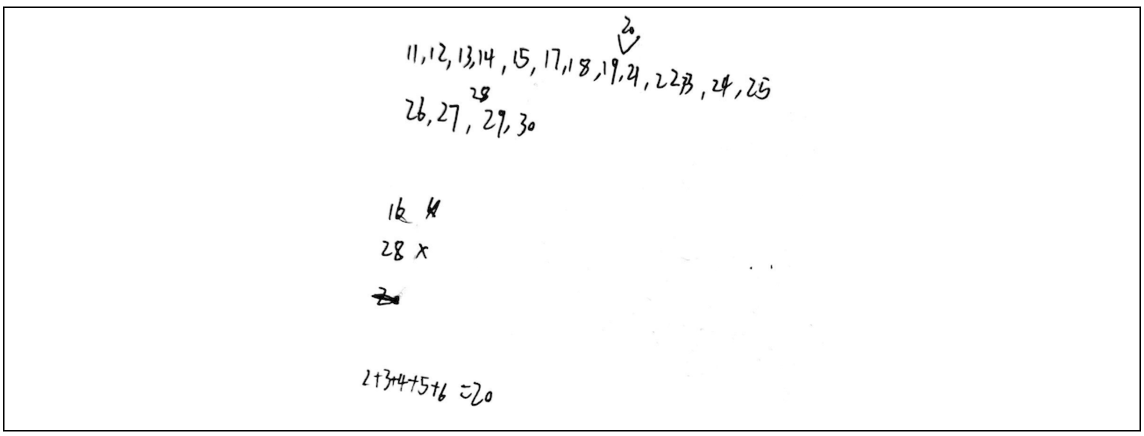


Figure 12. Andy's "Take Stock" check

Check and Reflect stage

The Uncertain Answer (UA) marker triggered in Andy a sequence of regulatory responses: systematic correction (SC), on-the-fly sub-goal checking (OC), subsequent error detection (ED), and ultimately self-explanation (SE).

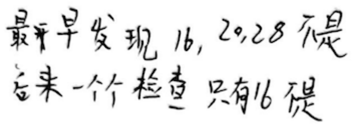
Andy's notes of UA	Translation
	At first I thought 16, 20, 28 were not nice numbers, but after checking one by one, only 16 is not

Figure 13. Andy's Uncertain Answer (UA) statement

Andy's note in the CR page (see Figure 13: "At first I thought 16, 20, 28 were not nice numbers, but after checking one by one, only 16 is not") suggests his awareness of the UA marker and this led to his systematic correction (SC). Specifically, drawing on his earlier parity reasoning, he tested the six-strings and eight-strings first, while temporarily setting aside the five- and seven-strings. He initially thought that even strings would produce even numbers. Here he over-generalized the idea that "the sum of an even number of consecutive integers is even." However, he appears to have recognized the mistake. In the six-strings case, he wrote down 1+2+3+4+5+6, but found it's odd sum and crossed it out (ED). For the eight-term sum, he calculated 1+2+…+8, which is 32 exceeded the 10-30 target range, and marked it with a line under the expression (OC). After this, he systematically returned to the odd-string cases which he earlier skipped: in the 5-strings case, he retained 2+3+4+5+6=20 and 4+5+6+7+8=30, while crossing out 3+4+5+6+7=25 as odd; in the 7-strings sum, he confirmed 1+2+…+7=28. He then undertook further ED by explicitly writing out and verifying previously skipped or uncertain expressions (e.g., see Figure 11: wrote down 9+10+11=30, 3+4+5+6+7, and 1+2+3+4+5 and then crossing out). Following this exhaustive checking, he finally got the answer that 16 was the only number between 10 and 30 that was not a nice number. To further justify his answer, Andy went beyond his solution and engaged in self-explanation (SE). From his solution, he abstracted a general principle and recorded in his worksheet: "for 3-string, nice number starting at 6, each differs by 3; for 4-string, nice numbers starting at 10, each differs by 4; for 5-string, starting at 15, each differs by 5; for n-string, each nice number differs by n" (see Figure 14).

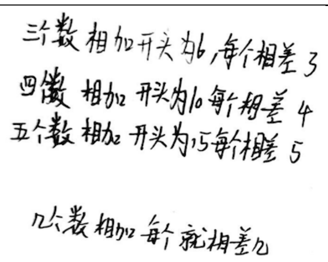
Andy's generalization	Translation
	• For 3-string, nice number starting at 6, each differs by 3; • for 4-string, nice numbers starting at 10, each differs by 4; • for 5-string, starting at 15, each differs by 5; • for n-string, each nice number differs by n

Figure 14. Andy's generalization Self Explanation (SE)

Summary

The case of Emma and Andy illustrates how the PSM worksheet supported different metacognitive problem solving pathways, but at the same time highlighted their metacognitive profiles. For Emma, her metacognitive profile was largely shaped by difficulties encountered in the early phases of problem solving. While other markers were present, her most salient responses were triggered by the understanding difficulty (UD) and inadequate progress (IP) red flags, which prompted regulatory actions that supported her continuation of the problem solving process. In contrast, Andy relied on an early key insight (KI) that shaped a comprehensive global strategy, allowing his work in the DM stage to proceed with fewer visible interruptions and with more local monitoring and responses. It was in the CR stage, when the uncertain answer (UA) red flag emerged, that Andy responded with a global systematic correction, supported by accompanying local regulation. Despite these different types of metacognitive profile, the worksheet provided both students with opportunities to engage in a metacognitively rich problem solving process and made their monitoring and regulation visible.

Discussion

The designed PSM worksheet provides a concrete integration of Schoenfeld's (1985) make-or-break points and Goos et al.'s (2000) metacognitive red flags within the structure of the Pólya model. Although challenges remain, the study shows positive signs. The students deliberately paused, reflected, and revised when they faced difficulty in understanding, inadequate progress, key insights or uncertainty about answers. Though small in scale, these findings suggest the potential feasibility of combining structured problem solving with explicit metacognitive monitoring and regulation at the theoretical level.

A methodological contribution of this study is the attempt to use a "marker–response" approach via the PSM worksheet to leave analyzable traces of students' work in a paper-and-pencil problem solving environment, thereby making certain on-line metacognitive processes more observable. Prior research has noted that metacognition is an internal process and is therefore often hidden and difficult to capture directly (Ng et al., 2021). Our approach attempts to analyze students' monitoring and regulation during problem solving by drawing on explicit metacognitive markers and corresponding metacognitive responses recorded on the worksheet. Compared with traditional think-aloud protocols, this approach is lower-cost, does not require video recording and transcription, and does not noticeably disrupt classroom instruction, while still enabling the tracing of routine and control metacognitive activities at key junctures. This attempt may offer a complementary way of examining the observability of certain on-line metacognitive processes in regular classroom settings.

This study also contributes toward developing a common language for talking about metacognition. This "common language" has the potential to make metacognition no longer an abstract or implicit process but a classroom resource that both teachers and students can use, discuss, and practice. It may help students transform metacognitive markers into adjustable, actionable nodes in problem solving, thereby supporting the targeting and effectiveness of teaching on metacognition.

Finally, the worksheet reported in this study is not merely a research tool, but a "concretisation" that can be used directly in the classroom. Teachers can incorporate it into routine instruction without additional technology or cumbersome preparation. At the same time, the worksheet is not fixed, but rather a practical tool. Following the principles of concretisation proposed by Leong et al. (2019), teachers may adapt the PSM worksheet to suit different instructional goals or problem contexts, but such adaptations should give attention to the principle of comportment. This means that the tool must preserve its core function of aligning metacognitive scaffolding with problem solving stages, so that classroom adaptations do not dilute its objectives. In this way, teachers' modifications can maintain both flexibility and fidelity, ensuring that the worksheet continues to support students' gradual internalisation of monitoring and regulation.

Conclusion

This study used a two-student case to show how metacognition can become visible during mathematical problem solving. The redesigned worksheet offers a simple way for students to mark their difficulties, notice key insights, adjust their strategies and confirm their answers. Given the small-scale nature of the case study, the findings reported here are not intended to support broad generalizations. Rather, they provide an illustrative account of how students' metacognitive monitoring and regulation can be made visible within a worksheet-structured problem solving setting. Although small in scale, the study suggests that the PSM worksheet may serve as a feasible and practical resource for integrating metacognitive attention into routine problem solving instruction. Future research could extend this work by examining more students, longer interventions, a wider range of tasks, or the roles of teachers' metacognition and students' collaborative problem solving in shaping metacognitively rich classroom environments.

References

Brousseau, G. (1997). *Theory of didactical situations in mathematics : didactique des mathematiques, 1970-1990* (1st ed. 2002.). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47211-2

China Ministry of Education. (2022). 义务教育数学课程标准 (2022 年版) [Compulsory education mathematics curriculum standards (2022 edition)]. Beijing Normal University Press.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45–75. https://doi.org/10.1007/s10649-005-0808-x

Dignath, C., & Büttner, G. (2008). Components of fostering self-regulated learning among students. A meta-analysis on intervention studies at primary and secondary school level. *Metacognition and Learning, 3*(3), 231–264. https://doi.org/10.1007/s11409-008-9029-x

Dong, F. (2021). 五年级小学生数学审题能力水平的调查及改进策略 [A survey and improvement strategies for fifth graders' problem comprehension in mathematics] (Master's thesis, Nanjing Normal University). https://doi.org/10.27245/d.cnki.gnjsu.2021.001682

Flavell, J. H. (1976). Metacognitive aspects of problem-solving. In L. B. Resnick (Ed.), *The nature of intelligence* (pp. 231–236). Lawrence Erlbaum Associates.

Flavell, J. H., Miller, P. H., & Miller, S. A. (2002). *Cognitive development* (4th ed.). Prentice-Hall.

Foong, P. Y. (2009). Problem solving in mathematics. In P. Y. Lee & N. H. Lee (Eds.), *Teaching primary school mathematics: A resource book* (pp. 54–81). Singapore: McGraw-Hill Education (Asia)

Garofalo, J., & Lester, F. K., Jr. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*(3), 163–176.

Goos, M., Galbraith, P., & Renshaw, P. (2000). A money problem: A source of insight into problem solving action. *International Journal for Mathematics Teaching and Learning*, 1–21.

Hatfield, L. L. (1978). Heuristical emphases in the instruction of mathematical problem solving: Rationales and research. In L. L. Hatfield & D. A. Bradbard (Eds.), *Mathematical problem solving: Papers from a research workshop* (pp. 21–42). Columbus, Ohio.

Henderson, K. B., & Pingry, R. E. (1953). Problem solving in mathematics. In H. F. Fehr (Ed.), *The learning of mathematics: Its theory and practice* (21st Yearbook of the National Council of Teachers of Mathematics) (pp. 228–270). Washington, DC: NCTM.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Information Age Publishing.

Holton, D., & Clarke, D. (2006). Scaffolding and metacognition. *International Journal of Mathematical Education in Science and Technology, 37*(2), 127–143.

Holton, D. A., Anderson, J., & Thomas, B. (1997). *OPEN plan for teaching mathematical problem solving.* Report No. 587, Ministry of Education, Wellington, New Zealand.

Kapur, M. (2011). A further study of productive failure in mathematical problem solving: Unpacking the design components. *Instructional Science, 39*(4), 561–579. https://doi.org/10.1007/s11251-010-9144-3

Kapur, M. (2014). Productive failure in learning math. *Cognitive Science, 38*(5), 1008–1022. https://doi.org/10.1111/cogs.12107

Kistner, S., Rakoczy, K., Otto, B., Dignath-van Ewijk, C., Büttner, G., & Klieme, E. (2010). Promotion of self-regulated learning in classrooms: Investigating frequency, quality, and consequences for student performance. *Metacognition and Learning, 5*(2), 157–171. https://doi.org/10.1007/s11409-010-9055-3

Kramarski, B. (2018). Teachers as agents in promoting students' SRL and performance: Applications for teachers' dual-role training program. In D. H. Schunk & J. A. Greene (Eds.), *Handbook of Self-Regulation of Learning and Performance* (2nd ed., pp. 223–239). Routledge. https://doi.org/10.4324/9781315697048-15

Lee, N. H. (2008). *Enhancing mathematical learning and achievement of secondary one Normal (Academic) students using metacognitive strategies.* [Doctoral dissertation, National Institute of Education, Nanyang Technological University]. https://hdl.handle.net/10497/2616

Lee, N. H., Yeo, D. J. S., & Hong, S. E. (2014). A metacognitive-based instruction for Primary Four students to approach non-routine mathematical word problems. *ZDM, 46*(3), 465–480. https://doi.org/10.1007/s11858-014-0599-6

Leong, Y. H., Tay, E. G., Quek, K. S., Toh, T. L., Toh, P. C., Dindyal, J., Ho, F. H., & Yap, R. A. S. (2014). *Making mathematics more practical: Implementation in the schools.* World Scientific.

Leong, Y. H., Tay, E. G., Toh, T. L., Quek, K. S., Toh, P. C., & Dindyal, J. (2016). Infusing mathematical problem solving in the mathematics curriculum: Replacement units. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems* (pp. 309–325). Springer International Publishing. https://doi.org/10.1007/978-3-319-28023-3_18

Leong, Y. H., Tay, E. G., Toh, T. L., Quek, K. S., & Yap, R. A. S. (2019). Concretisations: a support for teachers to carry out instructional innovations in the mathematics classroom. *International Journal of Science and Mathematics Education, 17*(2), 365–384. https://doi.org/10.1007/s10763-017-9868-5

Lester, F. K. (1983). Trends and issues in mathematical problem-solving research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*, 229–261.

Lester, F. K. (1989). *The Role of Metacognition in Mathematical problem-solving: A Study of Two Grade Seven Classes.* Final Report.

Lester, F. K. (1994). Musings about Mathematical Problem-Solving Research: 1970–1994. *Journal for Research in Mathematics Education, 25*(6), 660–675. https://doi.org/10.2307/749578

Lester, F. K. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast, 10*(1–2), Article 12. https://doi.org/10.54870/1551-3440.1267

Lester, F. K., & Kehle, P. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem-solving, learning, and teaching* (pp. 501–518). Lawrence Erlbaum.

Mevarech, Z. R., & Kramarski, B. (1997). IMPROVE: A Multidimensional Method for Teaching Mathematics in Heterogeneous Classrooms. *American Educational Research Journal, 34*(2), 365–394. https://doi.org/10.3102/00028312034002365

Ministry of Education, Singapore. (2021). *Mathematics syllabus: Primary one to six* (Implementation starting with 2021 Primary One cohort). Singapore: Author.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Authors. http://www.corestandards.org

Ng, K. E. D., Lee, N. H., & Safii, L. (2021). Facilitation of students' metacognition: Some insights gleaned from mathematics classrooms in Singapore secondary schools. In B. Kaur & Y. H. Leong (Eds.), *Mathematics instructional practices in Singapore secondary schools* (pp. 105–122). Springer Nature Singapore. https://doi.org/10.1007/978-981-15-8956-0_6

Nietfeld, J. L., Cao, L., & Osborne, J. W. (2005). Metacognitive monitoring accuracy and student performance in the postsecondary classroom. *The Journal of Experimental Education, 74*(1), 7–28.

Organisation for Economic Co-operation and Development. (2018). *The future of education and skills: Education 2030.* OECD Publishing. https://www.oecd.org/content/dam/oecd/en/publications/reports/2018/06/the-future-of-education-and-skills_5424dd26/54ac7020-en.pdf

Pólya, G. (1945). *How to solve it*. Princeton University Press.

Schraw, G., Crippen, K. J., & Hartley, K. (2006). Promoting self-regulation in science education: Metacognition as part of a broader perspective on learning. *Research in Science Education, 36*(1), 111–139. https://doi.org/10.1007/s11165-005-3917-8

Schraw, G., & Moshman, D. (1995). Metacognitive theories. *Educational Psychology Review, 7*(4), 351–371.

Schoenfeld, A. H. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive science, 7*(4), 329–363.

Schoenfeld, A. H. (1985). *Mathematical problem-solving.* Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). Macmillan.

Schoenfeld, A. H. (2013). Reflections on problem solving theory and practice. *The Mathematics Enthusiast, 10*(1–2), 9–34. https://doi.org/10.54870/1551-3440.1258

Schwartz, D. L., Chase, C. C., Oppezzo, M. A., & Chin, D. B. (2011). Practicing versus inventing with contrasting cases: The effects of telling first on learning and transfer. *Journal of Educational Psychology, 103*(4), 759–775. https://doi.org/10.1037/a0025140

Stein, M. K., Boaler, J., & Silver, E. A. (2003). Teaching mathematics through problem solving: Research perspectives. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grades 6–12* (pp. 245–256). NCTM.

Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *The Journal of Mathematical Behavior, 24*(3), 341–350. https://doi.org/10.1016/j.jmathb.2005.09.004

Veenman, M. V. J., Van Hout-Wolters, B. H. A. M., & Afflerbach, P. (2006). Metacognition and learning: Conceptual and methodological considerations. *Metacognition and Learning, 1*(1), 3–14. https://doi.org/10.1007/s11409-006-6893-0

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P.S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics*, 57–77.

Wilson, J., & Clarke, D. (2002). Monitoring mathematical metacognition. A paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA, April 1–5, 2002.

Zheng, Y. (2024). 小学生数学情境问题解决能力现状调查及提升策略研究 [Investigation and strategies to improve primary students' contextual problem-solving ability] (Master's thesis, Jiangxi Normal University). https://doi.org/10.27178/d.cnki.gjxsu.2024.001142

Zimmerman, B. J. (2008). Investigating self-regulation and motivation: Historical background, methodological developments, and future prospects. *American Educational Research Journal, 45*(1), 166–183. https://doi.org/10.3102/0002831207312909

Authors:
Shiqi JIN (NIE24.JS3752@e.ntu.edu.sg)
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616.

LEONG, Yew Hoong (yewhoong.leong@nie.edu.sg)
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616.