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The Remains of a Mathematics Problem

TAY Eng Guan, TOH Pee Choon, TOH Tin Lam, TONG Cherng Luen
National Institute of Education
Nanyang Technological University, Singapore

A problem first discussed in 2011 resurfaced when another colleague recently mentioned using it for a research project. On reacquainting with the problem, a comprehensive solution to some remains of the problem was obtained. This paper further postulates that it needed persistence of cognition and of effort to engender a reacquaintance and a resolve to find closure.

Keywords: problem solving, generalisations, persistence

Introduction

> Perhaps, then, there is something to his advice that I should cease looking back so much, that I should adopt a more positive outlook and try to make the best of what remains of my day. After all, what can we ever gain in forever looking back and blaming ourselves if our lives have not turned out quite as we might have wished? (Ishiguro, 1993, p. 244)

Unlike the fictional butler Stevens, from Ishiguro's novel *The Remains of the Day*, who mused about *not* looking back so as to make the best of the remains of his day, mathematicians, as per Pólya, often look back productively at the problems that they have solved. The authors too were drawn back to a previously solved problem when another colleague was overheard referring to the "nice numbers" in his new research project. Let us first state the problem.

Making Mathematics Practical (Toh et al., 2011) was a book written by a research project team for use as a teacher and researcher guide to implementing mathematical problem-solving à la Pólya (1954) in the secondary school mathematics curriculum. Problems 16 and 17 are as follows:

> **P16**. A 'nice' number is a number that can be expressed as the sum of a string of two or more consecutive positive integers. Determine which of the numbers from 50 to 70 inclusive are nice. (p. 129)
>
> **P17.** n is a positive integer that is not a power of 2. Show that n is a nice number. (p. 133)

An important feature of each problem was the possible adaptations, extensions and generalisations that were posed in the fourth phase of Pólya's Look Back (which we called Check and Expand). For Problem 17, these were as follows:

> *Adaptation:* Express 105 as the sum of the maximum number of consecutive integers.
>
> *Generalisation:* $n = p_1 p_2 \cdots p_k$, where the p_i's are distinct odd primes. What is the maximum number of consecutive integers for which n can be expressed as a sum of?
>
> *Extension:* A 'b-nice' number, where b is a positive integer, is a number that can be expressed as the sum of a string of two or more consecutive positive integers in at least b ways. Let $n = pq$, where p and q are odd primes. Show that n is a 2-nice number. (p. 134)

On checking the book, the authors realised that no solutions were provided for the adaptations, extensions, and generalisations. Although there was a faint recollection that these problems had been solved, the authors could not remember the solutions. Thus began a couple of weeks of work on solving the problems with the results detailed in the next section. We then end this article with some pedagogical considerations.

Characterising b-nice Numbers

In this section, we first recall some basic properties of nice numbers before we completely characterize b-nice numbers. The number 35 is a nice number because it can be expressed as $35 = 5 + 6 + 7 + 8 + 9$. Now 35 can also be expressed as $2 + 3 + 4 + 5 + 6 + 7 + 8$. These two expressions are clearly different because they involve different number of terms. In fact, it is impossible for two different strings of consecutive positive integers with the same number of terms to sum to the same number. For if the first term of one string is larger than that of the other string, every subsequent term would also be larger than the corresponding term in the other string.

Definition 1. Suppose a positive integer n can be expressed as the sum of a string of two or more consecutive positive integers $a + (a+1) + \cdots + (a+m-1)$ and also as the sum of another string of two or more consecutive positive integers $b + (b+1) + \cdots + (b+k-1)$. We say that the two strings are *different* if $m \neq k$.

Suppose a nice number n can be expressed as a consecutive string of m positive integers beginning from a, then we have

$$n = a + (a+1) + \cdots + (a+m-1) = \frac{1}{2}m(m+2a-1). \quad (*)$$

Since m and $m + 2a - 1$ are of different parity, we immediately deduce the following.

Proposition 1. If n is a nice number, then n must contain at least one non-trivial odd factor.

Two immediate consequences follow.

Corollary 1. If n is a power of 2, then it is not a nice number.

Corollary 2. If n is a nice number, then n can be expressed as $2^r p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where r is a non-negative integer, k is a positive integer, and for $i = 1, 2, \ldots, k$, p_i are distinct odd primes, s_i are positive integers.

Equation (*) can thus be rewritten as

$$2n = 2^{r+1}p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k} = m(m+2a-1).$$

By the Fundamental Theorem of Arithmetic, any factor of $2n$ must be of the form $2^{\beta_0}p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$, where β_j's are non-negative integers such that $0 \le \beta_0 \le r+1$ and $0 \le \beta_j \le s_j$ for each $j = 1,2,\dots,k$. A simple counting principle then implies that $2n$ has

$$(r+2)(s_1+1)\cdots(s_k+1)$$

distinct factors. However, not every factor can be a candidate for m (the number of terms in the string), since $1 < m < m+2a-1$. Furthermore, m and $m+2a-1$ are of different parity. This means that the factor 2^{r+1} must divide exactly one of m or $m+2a-1$. Hence, there are at most $2(s_1+1)\cdots(s_k+1)$ candidates for m. This number should be further halved as m is the smaller factor. Finally, since m is at least 2, we can exclude the trivial factor 1 to arrive at the following upper bound.

Proposition 2. Let $n = 2^r p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$, be a nice number, where r is a non-negative integer, k is a positive integer, and for $i = 1, 2, \dots, k$, p_i are distinct odd primes, s_i are positive integers. Then an upper bound for the number of different ways to express n as a sum of at least two consecutive positive integers is $(s_1+1)\cdots(s_k+1)-1$.

We shall show that the upper bound in Proposition 2 is exact. To illustrate our approach, let us reconsider the number $35 = 5 \times 7$ which can be expressed in two different ways as a string of consecutive positive integers:

$$5+6+\mathbf{7}+8+9 = 2+3+4+\mathbf{5}+6+7+8.$$

It can be observed that each of the factors 7 and 5, appear as the respective middle term in the strings. Furthermore, we can see that 6 and 8 can be paired to become two copies of 7 in the string 5 + 6 + 7 + 8 + 9. Likewise, 5 and 9 can be paired similarly to yield another two copies of 7. Thus, we can view this string as 5 groups of 7. Applying the same notion of pairing allows us to associate the string 2 + 3 + 4 + 5 + 6 + 7 + 8 as 7 groups of 5. This serves as a motivation to view a nice number $n = p \times q$ as p groups of q, and use its prime factorization to calculate exactly how many different ways n can be expressed as a string of consecutive positive integers. It is important to mention that 35 = 17 + 18 is another different representation and we shall explain later how this string fits into our notion of p groups of q. By appealing to Proposition 2, we see that no other representation is possible for 35.

Another observation from the three different ways to express 35 as a string of consecutive positive integers is that we only need to look at the last term in the string to decide if these strings are different. For if two strings have the same last term, they would also necessarily have the same second to last term, and thus inductively, all the terms are equal. This motivates the following result.

Proposition 3. Suppose a positive integer n can be expressed as the sum of a string of two or more consecutive positive integers $a+(a+1)+\cdots+(a+m-1)$ and also as the sum of another string of two or more consecutive positive integers $b+(b+1)+\cdots+(b+k-1)$. The two strings are *different* if and only if $(a+m-1) \neq (b+k-1)$.

To count how many different strings can represent nice numbers, it is more convenient to redefine b-nice numbers as follows:

Definition 2. A positive integer n is 'b-nice', where b is a positive integer, if it can be expressed as the sum of a string of two or more consecutive positive integers in *exactly* b different ways.

We provide a few examples: 1 and 2 are not nice numbers, 3 = 1 + 2 is a 1-nice number. 9 = 2 + 3 + 4 = 4 + 5 is 2-nice, and 21 = 1 + 2 + 3 + 4 + 5 + 6 = 6 + 7 + 8 = 10 + 11 is 3-nice. Our previous example of 35 is also 3-nice.

Now if we were to view 21 as 3 groups of 7, we obtain 6 + **7** + 8. However, if we were to view 21 as 7 groups of 3, we would end up with 0 + 1 + 2 + **3** + 4 + 5 + 6. The inclusion of 0 violates the requirement of strings of consecutive positive integers. This problem is further exacerbated when the number of groups is large. For example, viewing 33 as 11 groups of 3 would give

$$(-2) + (-1) + 0 + 1 + 2 + \mathbf{3} + 4 + 5 + 6 + 7 + 8.$$

A key insight by one of the authors is that this is not actually an issue because any string containing negative integers would necessarily also contain their respective additive inverses. Thus, this string will not be different from another string containing only positive terms. For example, $33 = -2 - 1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = (-2 + 2) + (-1 + 1) + 0 + 3 + 4 + 5 + 6 + 7 + 8 = 3 + 4 + 5 + 6 + 7 + 8$. We can now generalize Proposition 3 to allow the string to have nonnegative integer terms. The conclusion remains the same—the two strings are not different because they have the same last term.

Proposition 4. The sum of a string of m (≥ 2) consecutive positive integers, i.e., $a + (a + 1) + \cdots + (a + m - 1)$ is not different from the sum of the following string of $2a - 1 + m$ consecutive integers:

$$-(a-1) + \big(-(a-2)\big) + \cdots + 0 + \cdots + (a-1) + a + (a+1) + \cdots + (a+m-1).$$

Proposition 4 in the case of $m = 2$ means that

$$a + (a+1) = -(a-1) + \big(-(a-2)\big) + \cdots + 0 + \mathbf{1} + 2 + \cdots + (a-1) + a + (a+1),$$

where the latter can be viewed as $2a + 1$ groups of 1. (It is worthwhile to check that there are exactly a terms on each side of the middle term 1.) Since $a + (a + 1) = 2a + 1$, we can formalize this observation below.

Proposition 5. Let n be an odd integer greater than 2. Then it is possible to express n as the sum of a string of two consecutive positive integers associated with n groups of 1.

Proof.
The sum of a string of consecutive positive integers associated with n groups of 1 is

$$n = -\left(\tfrac{n-1}{2} - 1\right) + \cdots + 0 + \mathbf{1} + 2 + \cdots + \left(\tfrac{n-1}{2} - 1\right) + \tfrac{n-1}{2} + \tfrac{n+1}{2}.$$

By Proposition 4, this collapses into $\frac{n-1}{2} + \frac{n+1}{2}$. □

Thus, returning to our initial example of the 3-nice number 35 which can be factorized as $1 \times 35 = 35 \times 1 = 5 \times 7 = 7 \times 5$, we can associate each factorization to a string consisting of p groups of q. The exception is 1 group of 35. In other words, we claim that if $n = pq$ where p and q are odd primes, then n is a 3-nice number associated with p groups of q, q groups of p and pq groups of 1. Let us now formally define the idea of p groups of q.

Definition 3. Let $n = pq$, where p is an odd integer greater than 2, be a nice number. Then an expression of n as the sum of a string of consecutive positive integers associated with p groups of q is as follows:

$$\left(q - \frac{p-1}{2}\right) + \cdots + \boldsymbol{q} + \cdots + \left(q + \frac{p-1}{2}\right).$$

We remark that the representation associated with p groups of q does not necessarily contain p consecutive positive terms. However, any negative term that appears cancels with its additive inverse. (See Proposition 4.) In fact, when such cancellation occurs, an odd number of terms (including 0) is removed from the string, resulting in an even number of positive terms. For example, $33 = -2 - 1 + 0 + 1 + 2 + \mathbf{3} + 4 + 5 + 6 + 7 + 8 = 3 + 4 + 5 + 6 + 7 + 8$ is associated with 11 groups of 3 but it has 6 consecutive positive terms.

Proposition 6. Let $n = p_1 q_1 = p_2 q_2$, where p_1 and p_2 are distinct odd integers greater than 2, be a nice number. Then an expression of n as the sum of a string of consecutive positive integers associated with p_1 groups of q_1 is different from an expression of n as the sum of a string of consecutive positive integers associated with p_2 groups of q_2.

Proof.
The sum of a string of consecutive positive integers associated with p_1 groups of q_1 is $\left(q_1 - \frac{p_1-1}{2}\right) + \cdots + \boldsymbol{q_1} + \cdots + \left(q_1 + \frac{p_1-1}{2}\right)$. The sum of a string of consecutive positive integers associated with p_2 groups of q_2 is $\left(q_2 - \frac{p_2-1}{2}\right) + \cdots + \boldsymbol{q_2} + \cdots + \left(q_2 + \frac{p_2-1}{2}\right)$. We may assume that $p_1 < p_2$. Suppose otherwise that the two strings are not different. Then, by Proposition 3, we have $q_1 + \frac{p_1-1}{2} = q_2 + \frac{p_2-1}{2}$. Observe that $p_1 < p_2$ implies that $q_1 > q_2$. Hence, $q_1 - \frac{p_1-1}{2} > q_2 - \frac{p_2-1}{2}$ which means that the latter string is strictly longer than the former. As the strings are not different, Proposition 4 implies that $q_2 - \frac{p_2-1}{2} < 0$. Its corresponding additive inverse is $-\left(q_2 - \frac{p_2-1}{2}\right)$, which means that the smallest positive term in $\left(q_2 - \frac{p_2-1}{2}\right) + \cdots + \boldsymbol{q_2} + \cdots + \left(q_2 + \frac{p_2-1}{2}\right)$ after we have cancelled all the non-positive terms is $-\left(q_2 - \frac{p_2-1}{2}\right) + 1$.

Indeed, we must have $q_1 - \frac{p_1-1}{2} = -\left(q_2 - \frac{p_2-1}{2}\right) + 1$. Taking the difference of this equation with $q_1 + \frac{p_1-1}{2} = q_2 + \frac{p_2-1}{2}$ results in $p_1 = 2q_2 = \frac{2n}{p_2}$. In other words, $p_1 p_2 = 2n$, which is a contradiction since p_1 and p_2 are distinct odd integers.

Thus, an expression of n as the sum of a string of consecutive positive integers associated with p_1 groups of q_1 is different from an expression of n as the sum of a string of consecutive positive integers associated with p_2 groups of q_2. ☐

We can now completely characterize b-nice numbers.

Theorem 1. Let $n = 2^r m$, where r is a non-negative integer and m is an odd positive integer. Then we have the following.

a) If $m = 1$, then n is not a nice number.
b) Let n be a b-nice number. Let $m = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, where k is a positive integer, p_i, $i = 1, 2, \ldots, k$, are distinct odd primes, s_i, $i = 1, 2, \ldots, k$, are positive integers. Then

$$b = (s_1 + 1) \cdots (s_k + 1) - 1.$$

Proof.
Part a) follows from Corollary 1.

To prove part b), let $n = pq$ where p is an odd factor of n. Excluding the trivial factor $p = 1$, there are exactly $b = (s_1 + 1) \cdots (s_k + 1) - 1$ odd factors of n. For each of these odd factors, there exists an expression of n as the sum of a string of consecutive positive integers associated with p groups of q. Proposition 6 shows that these representations are all different. Since b equals the upper bound from Proposition 2, we know that there are exactly b different representations. ☐

Maximum Number of Consecutive Terms for Nice Numbers

While our definition of p groups of q for a nice number $n = pq$, allows us to completely characterize b-nice numbers, the integer p does not always equal the number of terms. In particular, when $n = 2a + 1$ is odd, Proposition 5 shows that the representation associated with n groups of 1 has exactly two consecutive positive terms, i.e., $n = a + (a + 1)$. We can build on this idea to decompose odd integers into pairs, which we shall call dyads. For example, $36 = 4 \times 9$. We can decompose 9 into four dyads, namely $4 + 5, 3 + 6, 2 + 7, 1 + 8$, resulting in the representation $36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.

Definition 4. Let $n = pq$, where p is an odd integer greater than 2. Then an expression of n as the sum of a string of consecutive positive integers associated with q dyads of p is as follows:

$$\left(\frac{p-1}{2} - q + 1\right) + \cdots + \left(\frac{p-1}{2} - 1\right) + \boldsymbol{\frac{p-1}{2}} + \left(\boldsymbol{\frac{p-1}{2} + 1}\right) + \left(\frac{p-1}{2} + 2\right) + \cdots + \left(\frac{p-1}{2} + q\right).$$

Proposition 3 shows that the representation associated with q dyads of p is not different from the representation associated with p groups of q. For example, $18 = 3 \times 6$ can be associated with 3 groups of 6, as well as 6 dyads of 3 but the two representations

$$5 + \mathbf{6} + 7 = -4 + \cdots + 0 + \mathbf{1} + \mathbf{2} + 3 + 4 + 5 + 6 + 7$$

are not different as one involves non-positive terms. The introduction of dyads allows us to refer to the representation consisting of positive terms depending on the relative size of the factors. When $q > \frac{p-1}{2}$, the representation associated with p groups of q contains p consecutive positive terms. On the other hand, when $q \le \frac{p-1}{2}$, the representation associated with q dyads of p contains $2q$ consecutive positive terms. It is worth mentioning that special cases of each of the representations given in Definitions 3 and 4 can be found in Toh et al. (2011, p.134) but the definitions of groups and dyads are new.

Let us illustrate with the example, $27 = 3 \times 9$ which is a 3-nice number. The three representations are associated with 3 groups of 9, 3 dyads of 9 (instead of 9 groups of 3) and 1 dyad of 27 (instead of 27 groups of 1), i.e.,

$$8 + \mathbf{9} + 10 = 2 + 3 + \mathbf{4} + \mathbf{5} + 6 + 7 = \mathbf{13} + \mathbf{14}.$$

We now turn to the problem of finding the maximum number of terms in the string.

Definition 5. If n is a nice number, we define M_{odd} as the maximum number of terms in a string of odd consecutive positive integers that adds up to n. Similarly, M_{even} is the maximum number of terms in a string of even consecutive positive integers that adds up to n.

Theorem 2. Let $n = 2^r p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$, be a nice number, where r is a non-negative integer, k is a positive integer, and for i = 1, 2, …, k, p_i are distinct odd primes, s_i are positive integers. Then,

$$M_{\text{odd}} = \max\left\{\prod_{i=1}^{k} p_i^{t_i} \,|\, 0 \le t_i \le s_i, \prod_{i=1}^{k} p_i^{t_i} < \sqrt{2n}\right\},$$
$$M_{\text{even}} = \max\left\{2^{r+1} \prod_{i=1}^{k} p_i^{s_i - t_i} \,|\, 0 \le t_i \le s_i, 2^{r+1} \prod_{i=1}^{k} p_i^{s_i - t_i} < \sqrt{2n}\right\},$$

and the maximum number of terms in a string of consecutive positive integers that adds up to n is the maximum of M_{odd} and M_{even}.

Proof.
Let a string of consecutive positive integers that adds up to n be $a + \cdots + (a + m - 1)$, i.e., the first term is a and the number of terms is m. Then, we have $2n = m(2a + m - 1)$, where m and $2a + m - 1$ have opposite parities. Since $m < 2a + m - 1$, we have $m < \sqrt{2n}$. The expression for m then follows from its parity and the prime factorization of n. ☐

We illustrate the theorem with the example $180 = 2^2 \times 3^2 \times 5$. M_{odd} is the largest odd factor of $180 < \sqrt{360} \approx 18.97$. So $M_{\text{odd}} = 3^1 \times 5^1 = 15$. This is the string associated with 15 groups of 12, i.e., $5 + \cdots + \mathbf{12} + \cdots + 19$. For even m, M_{even} is the largest factor of the form $8j$ of 360, where j is odd and $8j < 18$, which means $M_{\text{even}} = 2^3 \times 3^0 \times 5^0 = 8$. This is the string associated with 4 dyads of 45, i.e., $19 + \cdots + \mathbf{22} + \mathbf{23} + \cdots + 26$. Clearly, the maximum number of terms in a string of consecutive positive integers that adds up to 180 is 15.

Corollary 3. Let $n = p_1 p_2$, where $p_1 < p_2$ are distinct odd primes. If $2p_1 > p_2$, then the maximum number of terms in a string of consecutive positive integers that add up to n is p_2. Otherwise, it is $2p_1$.

Proof.
The possible candidates for M_{odd} are p_1 or p_2. The criteria $p_i < \sqrt{2p_i p_j}$ is equivalent to $p_i < 2p_j$ where $i \ne j$. So if $p_2 < 2p_1$, then $M_{\text{odd}} = p_2$ and at the same time $M_{\text{even}} = 2$. However, if $2p_1 < p_2$, then $M_{\text{even}} = 2p_1$ and $M_{\text{odd}} = p_1$. ☐

We illustrate Corollary 3 with two examples, $n = 35$ and $n = 85$. When $n = 35 = 5 \times 7$, since $2 \times 5 > 7$, the maximum number of terms in a string of consecutive positive integers is 7. This is associated with 7 groups of 5. The other possible representations of 35 is 5 groups of 7, and 1 dyad of 35 (i.e., $17 + 18$), yielding 5 and 2 positive terms respectively. On the other hand, when $n = 85 = 5 \times 17$, since $2 \times 5 < 17$, Corollary 3 shows that the maximum number of terms is 10. This corresponds to 5 dyads of 17, namely

$$4 + 5 + 6 + 7 + \mathbf{8} + \mathbf{9} + 10 + 11 + 12 + 13.$$

The other representations are associated with 5 groups of 17 and 1 dyad of 85.

We now extend our analysis to a nice number of the form $n = p_1p_2p_3$, where $p_1 < p_2 < p_3$ are distinct odd primes. Table 1 lists all the possible odd factors for n and the conditions for that factor to be considered as M_{odd}. Likewise, candidates for M_{even} are listed in Table 2.

Table 1.
Odd factors of $n = p_1p_2p_3$ *as candidates for* M_{odd}

Candidates for M_{odd}	**Equivalent condition to $\prod_{i=1}^{k} p_i^{t_i} < \sqrt{2n}$**
p_1	$p_1 < 2p_2p_3$ (always holds)
p_2	$p_2 < 2p_1p_3$ (always holds)
p_3	$p_3 < 2p_1p_2$
p_1p_2	$p_1p_2 < 2p_3$
p_1p_3	$p_1p_3 < 2p_2$ (never holds)
p_2p_3	$p_2p_3 < 2p_1$ (never holds)
$p_1p_2p_3$	$p_1p_2p_3 < 2$ (never holds)

Table 2.
Even factors of $2n = 2p_1p_2p_3$ *as candidates for* M_{even}

Candidates for M_{even}	**Equivalent condition to $2\prod_{i=1}^{k} p_i^{t_i} < \sqrt{2n}$**
2	$2 < p_1p_2p_3$ (always holds)
$2p_1$	$2p_1 < p_2p_3$ (always holds)
$2p_2$	$2p_2 < p_1p_3$ (always holds)
$2p_3$	$2p_3 < p_1p_2$
$2p_1p_2$	$2p_1p_2 < p_3$
$2p_1p_3$	$2p_1p_3 < p_2$ (never holds)
$2p_2p_3$	$2p_2p_3 < p_1$ (never holds)
$2p_1p_2p_3$	$2p_1p_2p_3 < 1$ (never holds)

By considering various ranges of values for p_3, we can deduce M_{odd} and M_{even}. For example, if $p_3 > 2p_1p_2$, then $M_{\text{odd}} \neq p_3$ but since $p_1p_2 < 2p_3$ holds, $M_{\text{odd}} = p_1p_2$. At the same time, $M_{\text{even}} \neq 2p_3$ but $M_{\text{even}} = 2p_1p_2$ holds. Other cases can be argued in a similar manner and we record the outcome as a corollary.

Corollary 4. Let $n = p_1p_2p_3$, where $p_1 < p_2 < p_3$ are distinct odd primes, then the following holds:

a) If $p_3 > 2p_1p_2$, then $M_{\text{odd}} = p_1p_2 < M_{\text{even}} = 2p_1p_2$;

b) If $p_1p_2 < p_3 < 2p_1p_2$, then $M_{\text{odd}} = p_3 > M_{\text{even}} = 2p_2$;

c) If $\frac{p_1p_2}{2} < p_3 < p_1p_2$, then $M_{\text{odd}} = p_1p_2 > M_{\text{even}} = 2p_2$;

d) If $p_3 < \frac{p_1p_2}{2}$, then $M_{\text{odd}} = p_3 < M_{\text{even}} = 2p_3$.

Summary for Pedagogical Considerations

三上 (Sān shàng, n.d.) translated as “Three Upons", was written by Ouyang Xiu (c. 1000). The “Three Upons" are explicated as 马上 (mǎ shàng), 枕上 (zhěn shàng), 厕上(cè shàng), translated as “upon the horse, upon the pillow, upon the toilet”. The aphorism emphasizes that learning (which we apply to solving a problem) requires making good use of fragmented time and maintaining concentration (ibid.). The three scenarios indicate thinking about the problem while travelling, while lying on the bed and while easing oneself! The authors and their colleagues have experienced working on mathematics problems in such scenarios. One of them, a well published researcher in graph theory remarked that he once solved a problem when he was riding his bicycle. When questioned about the other two scenarios, he said yes to solving problems while resting on the bed but with regard to the third, somewhat true in that he gained some insight when he left his desk to ease himself. Relooking at the nice number problem, the first author worked on it “upon the horse” while driving, “upon the pillow” quite often, but not “upon the toilet seat”, having been told that it is medically inadvisable to be seated there for a long time.

Indeed, with regard to the problem of nice numbers, the remains of the earlier problem had persisted in the minds of the authors over more than a decade. We surmise that the name “nice numbers” also play an important role. While the name “nice numbers” does not carry any mathematical meaning, it provides a useful reference for the authors each time they revisited the problem in various classes that they had taught to students at various levels over the years. When stirred to action, there was little inertia because the problem had persisted in the cognition even after many years. What was needed was an added persistence in effort which was easily generated because of the interesting nature of the problem.

Pedagogically, teachers should encourage persistence in their students. Persistence in cognition can be encouraged by posting reminders of unsolved problems on the class noticeboard or verbal reminders in the course of lessons through the year. Persistence in effort should be encouraged as a counter to the teacher providing the answers all the time. To this end, difficult problems which the teacher herself has not solved and is trying to solve are platforms for mutual learning and effort. Perhaps you would like to extend Corollary 4 to the case of $n = p_1p_2p_3p_4$ or try the following adaptation:

> An ‘oddly-nice’ number is a number that can be expressed as the sum of a string of two or more consecutive odd positive integers. Determine which of the numbers from 50 to 70 inclusive are oddly-nice.

We would also like to add into the literature (see for example, Ho et al., 2023) our experience of using Excel to explore a problem to form conjectures. In this episode, we formulated Corollaries 3 and 4 by studying many examples on Excel. The visual affordance of the spreadsheet supersedes any graphing calculator or handwritten notes.

On further reflection, we could see how the preciseness of definitions led to easier communication in problem-solving discussion and in the final writing of the paper. *Definition* is stated as a Big Idea in mathematics for the Singapore school mathematics curriculum (see for example, Tay, 2019). In this episode, we changed the definition of b-nice from "at least b representations" to "exactly b representations". The tighter definition allowed a complete characterization of all positive integers, i.e., Theorem 1, which would not have been possible with the original definition which did not divide the positive integers into non-intersecting classes.

The concept of "groups" came from the well-known school method of "grouping" terms in an arithmetic progression. Although the "grouping" for an odd sequence is different from that of an even sequence, the initial 'definition' by Theo (pseudonym of one of the authors) was:

> Let a group of a positive integer x be a sum of two integers a and b that add up to x, or x itself. A sum of two integers a and b that add up to $2x$ is considered as two groups of x.

While the 'definition' took care of the two possible ways of "grouping", the two statements in the 'definition' were contradictory and earned the ire of Peter (pseudonym of another author) who constantly badgered Theo to uphold the rigour of formal mathematics communication. It is to Theo's credit that he took the umbrage positively and conceived of the final definitions of group and dyad. Indeed, these definitions made clear that each factorization of n into pq, where p is odd, is associated with two sequences: p groups of q and q dyads of p, with exactly one sequence consisting entirely of positive integers.

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Authors:
TAY, Eng Guan (engguan.tay@nie.edu.sg),
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616

TOH, Pee Choon (peechoon.toh@nie.edu.sg),
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616

TOH, Tin Lam (tinlam.toh@nie.edu.sg),
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616

TONG, Cherng Luen (chernglueng.tong@nie.edu.sg),
National Institute of Education, Nanyang Technological University, Singapore,
1 Nanyang Walk, Singapore 637616