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From Euclidean Geometry to Euclidean Spaces

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Kronecker famously said "God made the integers, all else is the work of man." Here is a sketch of how Euclidean spaces are motivated by Euclidean geometry, which is viewed explicitly as a science. The unusual approach highlights the abstractness and hence utility of Euclidean spaces, and offers an intuitive interpretation of the Cauchy-Schwarz inequality.

Keywords: Euclidean geometry, coordinate geometry, Euclidean space, Cauchy-Schwarz inequality

Introduction

The axiomatic treatment of geometry in Euclid's *Elements* has been spectacularly influential in the instruction of reasoning for millenia, despite its logical flaws. Euclidean geometry is the motivation for coordinate geometry, and the more general theory of Euclidean spaces. By a Euclidean space we mean the set $\mathbb{R}^n$, n a positive integer, whose elements are called points. Let the points $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ be labelled A and B. The point $(a_1 + b_1, \dots, a_n + b_n)$ is labelled $A + B$, and for a real number k, $(ka_1, \dots, ka_n)$ is labelled kA. But of more interest here are two definitions: (i) the distance between points A and B is $\sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$; (ii) A and B are orthogonal if the dot product $A \cdot B = a_1 b_1 + \cdots + a_n b_n = 0$; which are rooted in Euclidean geometry. The connection may be described in high school coordinate geometry, while undergraduate textbooks on linear algebra typically make only brief references to Euclidean geometry after defining the Euclidean spaces (Lang, 1971; Strang, 2023). Thus, here is an opportunity for an exposition at the undergraduate level. Moreover, in order to appreciate the full significance of the connection, we view Euclid's propositions not as abstract theorems, but as natural laws: statements governing our physical world. In other words, like mechanics and probability, geometry is a science (Feller, 1968; Goodman, 1979; von Mises, 1957), probably the first mathematical science. We will go from two scientific facts, on distance between two points and on orthogonal line segments, to mathematical definitions. The journey will show clearly that the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are more general than merely mathematical models of the plane or space which is so familiar to us. Moreover, an examination of angle and orthogonal projection further illustrates the fact-definition conversion and sheds intuitive light on the Cauchy-Schwarz inequality. It is hoped that this perspective helps the student pivot more meaningfully from Euclidean geometry to Euclidean spaces, and that Euclid might have found it useful for this purpose.

Facts in Euclidean Geometry

We first set up the coordinates in Euclid's plane. Although trite, it deserves a deliberate refresher. Using Euclidean geometry we can construct perpendicular lines, which extended indefinitely are the horizontal and vertical axes. This is a concrete, scientific fact, not an abstract, mathematical proposition. Let the intersection point be the origin O with coordinates (0,0). Given a unit length, a point A has unique coordinates (a_1, a_2), where $|a_1|$ and $|a_2|$ are respectively the horizontal and vertical distances between O and A.

Let $A(a_1, a_2)$ and $B(b, b_2)$ be two points such that the line segment AB is not parallel to either axis, i.e., $a_1 \neq b_1$ and $a_2 \neq b_2$. Let C have coordinates (b_1, a_2). Since the axes are perpendicular, $\angle ACB = 90^\circ$, and by the Pythagorean Theorem (Proposition 47, Euclid), the distance between A and B is $|AB| = \sqrt{|AC|^2 + |BC|^2} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$. Clearly, this formula also works if AB is horizontal or vertical, or even if $A = B$. Thus we have the following fact.

F1(2) The distance between two points $A(a_1, a_2)$ and $B(b, b_2)$ in the plane is

$$|AB| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}.$$

Suppose $\angle AOB = 90^\circ$. Without loss of generality, let A be clockwise from B. If A is in the first quadrant, i.e., $a_1 > 0$ and $a_2 > 0$, B must be in the second quadrant: $b_1 < 0, b_2 > 0$. Let points P and Q have coordinates $(a_1, 0)$ and $(0, b_2)$. Since $\triangle OPA$ and $\triangle OQB$ are similar, $|PA|/|QB| = |OP|/|OQ|$, i.e., $a_2 b_2 = a_1(-b_1)$, or $a_1 b_1 + a_2 b_2 = 0$. The same conclusion is obtained if A is in the second, third $(a_1 < 0, a_2 < 0)$ or fourth quadrant. It also holds if A and B lie on the axes. The converse can be obtained by reversing the arguments.

F2(2) Let $A(a_1, a_2) \neq O$ and $B(b, b_2) \neq O$ be two points in the plane. $OA \perp OB$ if and only if $a_1 b_1 + a_2 b_2 = 0$.

Next, we deal with space. Set up three mutually perpendicular axes, so that every point is specified by three numbers. Suppose points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are such that $a_i \neq b_i$ for $i = 1, 2, 3$. Let C and D have coordinates (b_1, a_2, a_3) and (b_1, b_2, a_3). Then $\angle ACB = 90^\circ$, implying $|AB|^2 = |AC|^2 + |CB|^2$. Also, $\angle CDB = 90^\circ$, so that $|CB|^2 = |CD|^2 + |DB|^2$. Therefore, $|AB|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$. If for some i, $a_i = b_i$, A and B can be treated as points in the plane, and F1(2) gives the same conclusion.

F1(3) The distance between two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ in space is

$$|AB| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}.$$

Let O, A, B be different points. They are collinear if they lie on a straight line. Suppose OAB is a triangle, so that O, A, B are not collinear. By expanding $|AB|^2$, we get

$$|AB|^2 = |OA|^2 + |OB|^2 - 2(a_1 b_1 + a_2 b_2 + a_3 b_3).$$

Like in the plane, by the Pythagorean Theorem and its converse, $OA \perp OB$ exactly if $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$. If O, A, B are collinear, by considering similar triangles, we know that $a_i = k b_i$ for $i = 1,2,3$, where $k > 0$ if A and B are on the same side of O, and $k < 0$ if O is between A and B. Then $a_1 b_1 + a_2 b_2 + a_3 b_3$ is strictly positive and strictly positive in these two cases.

F2(3) Let $A(a_1, a_2, a_3) \neq O$ and $B(b_1, b_2, b_3) \neq O$ be two points in space. $OA \perp OB$ if and only if $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

The argument of F2(3) also works for F2(2).

Definition of Euclidean Spaces

We turn the facts in the previous section into definitions. Let $A(a_1, \dots, a_n), B(b_1, \dots, b_n) \in \mathbb{R}^n$. Their dot product is $A \cdot B = a_1 b_1 + \cdots + a_n b_n$. The distance between them is $\sqrt{(B - A) \cdot (B - A)} = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$, abbreviated as $|B - A|$. Also, $|A - O|$ is abbreviated as $|A|$. If $A \cdot B = 0$, A and B are orthogonal, written $A \perp B$.

The Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are respectively mathematical models of the plane and space: the definitions of distance and orthogonality match geometric facts. $\mathbb{R}^1$ is a model of a line, although here orthogonality is an odd idea. For $n \geq 4$, $\mathbb{R}^n$ is not a model of a real "hyperspace", in the sense that the definitions are not backed up by empirical facts. For instance, given $A(a_1, a_2, a_3, a_4)$ and $B(b_1, b_2, b_3, b_4)$ in $\mathbb{R}^4$ where $a_i \neq b_i$ for $i = 1, 2, 3, 4$, let C have coordinates (b_1, b_2, b_3, a_4). Then we do not know on empirical ground that $\angle ACB = 90^\circ$; rather it follows from $(a_1 - b_1, a_2 - b_2, a_3 - b_3, 0) \perp (0,0,0, b_4 - a_4)$. Despite $\mathbb{R}^4, \mathbb{R}^5, \dots$, being unreal in this sense, the algebra makes it possible to reason about lines, planes and solid figures as if they are in the plane or space. Thus geometric intuition is very helpful for studying Euclidean spaces.

An element of $\mathbb{R}^n$ is also called a vector, which in the plane or space looks like an arrow from the origin to the point. Since $\mathbb{R}^n$ is closed under coordinate-wise addition, mathematicians declare that the sum of two vectors must be a vector. This forces $O = (0, \dots, 0)$ to be a vector, the "zero vector", even though it does not look like an arrow.

Suppose $A \neq O$, $B \neq O$, and $A \perp B$. For any nonzero k, $A \perp kB$. In the plane or space, this means the line containing OB is perpendicular to OA. However, $A \perp kB$ also holds for $k = 0$, although it makes no sense for a point to be perpendicular to a line. The mathematical model has extraneous elements relative to the real thing, which one should watch out for when translating theory into practice.

Unlike the establishment of the facts in the previous section, the n-dimensional Euclidean space does not require axes. For $n = 2$, the horizontal and vertical axes can be defined as the subsets $\{(a_1, 0) : a_1 \in \mathbb{R}\}$ and $\{(0, a_2) : a_2 \in \mathbb{R}\}$. They are orthogonal by definition, not by construction. Consider a plane where the "positive vertical axis" makes a 45° angle with the "positive horizontal axis". Put the nonnegative integers at equally spaced marks along the axes, and let points A, B, C have coordinates $(1,0), (0,1), (1,1)$. Then by definition $OA \perp OB$ and $|AB| = |OC|$, even though these look wrong. A similar discordance arises if the angle between the axes is any value other than 90°, but this liberation of the Euclidean spaces is fine, and enables it to be applied to new problems. A stranger example: every adult in a city is a point, where the first coordinate is the difference between the adult's height and the average height of the city, and the second coordinate is the net asset of the adult. Then an adult of average height is orthogonal to an adult whose net asset is \$0, even though the substantive meaning of this statement is unclear. We can also talk about distance and orthogonality in a Euclidean space where the points are pubs and the coordinates are the numbers of tables, chairs and beer mugs, which recalls Hilbert's famous remark (Reid, 1996) on his axiomatisation of Euclidean geometry. In summary, more than affirming the algebraic character of Euclidean geometry, the Euclidean spaces unleash the power of algebra on numerous practical problems, of which machine learning is an excellent example.

Angles in Euclidean Spaces

In the plane or space, the cosine rule says: If OAB is a triangle, $|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\angle AOB$. Compare with the displayed equation just before F2(3) or the two-dimensional counterpart, and we get $A \cdot B = |A||B|\cos\angle AOB$. Can we turn this fact around into a definition of angle in $\mathbb{R}^n$: If $A \neq O$ and $B \neq O$, $\angle AOB$ is $\cos^{-1}\left(\frac{A\cdot B}{A\cdot A}\right)$? It works in the case $A \perp B$, since $\frac{A\cdot B}{A\cdot A} = 0$ regardless of how large $|A||B|$ is. Otherwise we need to know that the argument of the inverse cosine is always between -1 and 1. The assurance comes from a famous result.

Cauchy-Schwarz Inequality For any $A, B \in \mathbb{R}^n$, $|A \cdot B| \leq |A||B|$, and equality holds if and only if $A = kB$ for some $k \in \mathbb{R}$.

Clearly, equality holds if either $A = O$ or $B = O$, or otherwise if A and B are collinear, i.e., $A = kB$ for some $k \in \mathbb{R}$. Hence, the following statement is equivalent:

Let $A, B \in \mathbb{R}^n$. If they are not collinear, $|A \cdot B| < |A||B|$. Otherwise, equality holds.

To prove the above, we use the idea of dropping a perpendicular (Cannon, 1989), which is an excellent illustration of the Euclidean connection. In the plane or space, let OAB be a triangle such that $\angle AOB \neq 90^\circ$. Then there is a unique point P on the line containing OA such that $OP \perp BP$. BP is the perpendicular dropped from OB on OA, and OP is the orthogonal projection of OB on OA. It is straightforward to deduce the following from the previous facts.

F3 In the plane or space, let OAB be a triangle such that $\angle AOB \neq 90^\circ$. The perpendicular dropped from OB to OA is BP, where $P = \frac{A\cdot B}{A\cdot A}A$.

Like before, we turn F3 into a definition. In $\mathbb{R}^n$, suppose A and B are not collinear, and $A \cdot B \neq 0$. Then $P = \frac{A\cdot B}{A\cdot A}A$ is the orthogonal projection of OB on OA. Now $P \neq O$, $B - P \neq O$, and $P \perp (B - P)$. Hence $|B|^2 = |P|^2 + |B - P|^2 > |P|^2$, i.e., $|A \cdot B| < |A||B|$. If $A \perp B$, we define $P = O$ and get the same conclusion. Thus, the Cauchy-Schwarz Inequality says the hypotenuse of a right triangle in $\mathbb{R}^n$ cannot be shorter than another side. This intuitive interpretation is another gift from the science of Euclidean geometry, and is arguably more insightful than a purely algebraic proof such as in (Ziegler et al., 2018).

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