

Developing Algebraic Thinking at Primary Levels

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Algebraic thinking encompasses representing, generalising, and formalising patterns and regularity in mathematics. This paper reviews definitions of algebraic thinking, discusses ways of developing algebraic thinking using generalisation and problem-solving approach. A generalising task and an algebraic word problem with solutions representations are highlighted to show how algebraic thinking could be developed at the primary levels.

Keywords: Algebraic Thinking, Generalisation, Problem-solving approach

Introduction

Miss Wong, a mathematics teacher, showed the following algebraic word problem to her primary five pupils during a mathematics lesson:

Ann made some tarts to sell. There were 62 more pineapple tarts than egg tarts. After 146 pineapple tarts were sold, the number of egg tarts was 5 times as many as the number of pineapple tarts. How many egg tarts were there at first?

She first asked her pupils for ideas on how they could represent the relationships between quantities and wrote their ideas on the board. The pupils suggested drawing a picture, using square paper, making a table, drawing models, guess and check as well as explaining in words. Miss Wong then told her pupils to represent the problem in different ways such that one of the approaches illuminated a visual or symbolic representation. This short preview into Miss Wong's mathematics lesson shows the importance of analysing relationships between quantities. She encouraged her pupils to suggest different solutions representations before they even commenced to solve the algebraic word problem, and then she proceeded to let the pupils to decide for themselves which solution representations they wanted to use to represent the context of the algebraic problem. This algebraic word problem has initial unknowns, and it is algebraic as the problem-solving route starts with unknown quantities (Bednarz & Janvier, 1996): We know neither the number of pineapple tarts nor the number of egg tarts owned by Ann. Such an algebraic word problem requires algebraic thinking and forming algebraic linear equations could be a challenge for primary five pupils. Moreover, many primary school pupils are not able to solve such algebraic word problems as such problems require knowledge of letter-symbolic algebra as well as the ability to construct and solve a system of linear equations with up to two unknowns.

Therefore, the main purpose of this paper is to discuss some perspectives of algebraic thinking and review two approaches used by the Singapore primary mathematics curriculum to develop algebraic thinking. The paper also exemplifies two distinct aspects of algebraic thinking that can be identified in primary mathematics instruction: generalising and specialising as well as analysing parts and whole. These two components of algebraic thinking provide a useful framework for teaching primary pupils to think algebraically and for determining whether a problem can be viewed algebraically.

Algebraic Thinking

Radford (2006) argues that “using letters does not amount to doing algebra” (p. 3). He establishes that, primarily, algebraic thinking is a distinct way of reflecting mathematically and that the use of alphanumeric symbols is not essential; other semiotic representations may also be used. The need for enriching primary mathematics through the integration of algebra has led to numerous research studies that contributed to the identification of certain characteristics of algebraic thinking at the primary levels (Blanton et al., 2011; Kieran, 2007; Radford, 2014).

What is algebraic thinking at the primary levels? While there is no clear consensus about the answer to this question, there is a general concurrence that algebraic thinking at the primary levels goes beyond arithmetic and computational fluency and focuses on the fundamental structure of mathematics (Cai & Knuth 2005). Algebraic thinking encompasses representing, generalising, and validating patterns and regularity in all aspects of mathematics (Van de Walle, 2004). We can interpret this definition to mean that algebraic thinking is about generalising arithmetic operations and operating on unknown quantities. It involves recognising and analysing patterns and developing generalisations about these patterns. Another definition of algebraic thinking is the “capacity to represent quantitative situations so that relations among variables become apparent” (Steele & Johanning, 2004. p.65). Blanton and Kaput (2005) defined algebraic thinking as a habit of mind that allows pupils to identify and express mathematical structure and relationships, such as the structure in arithmetical and symbolic expressions (Radford, 2003), the relationships in numerical and geometrical patterns (Mulligan & Mitchelmore, 2009; Warren & Cooper, 2008). The development of algebraic thinking at the primary levels demands the development of specific ways of thinking that result from analysing relationships between quantities, noticing structure, studying change, generalising, problem solving, modeling, justifying, proving, and predicting (Kieran 2004). When pupils think algebraically when solving problems, various habits of thinking come into play, such as doing—undoing, building rules to represent functions, and abstracting from computation (Driscoll, 1999).

Blanton & Kaput (2005) further explained that algebraic thinking requires four competencies: (1) understanding patterns, relations, and functions; (2) representing and analysing mathematical situations and structures using algebraic symbols; (3) using mathematical models to represent and understand quantitative relationships; and (4) analysing changes in various contexts. More recently, Radford (2014) expressed algebraic thinking as a mode of mathematical reflection and action, emphasising the need to identify fundamental forms of algebraic thinking that are not exclusively based on alphanumeric symbolism and the numerical and geometrical structure in tables, graphs, and number lines. Radford further elaborated that there are three basic requirements that exemplify algebraic thinking: (1) indeterminacy, the problem involves unknown quantities; (2) denotation, the unknown quantities are showed in

various ways which are not only limited to alphanumeric symbols; and (3) analyticity, the unknown quantities are added, subtracted, multiplied or divided, as if they were known.

Analysis of the Singapore primary mathematics curricular materials provided evidence to show that two approaches to develop algebraic thinking are adopted in the Singapore primary mathematics curriculum; these are generalisation and problem solving (Ng, 2004). Ng further elaborated that these two approaches are supported by thinking processes emphasised in the Singapore primary mathematics curriculum — generalising and specialising as well as analysing parts and whole respectively. The generalisation approach and problem-solving approach are discussed in this paper.

Developing Algebraic Thinking using Generalisation Approach

Generalisation is a central component of algebraic thinking. It can take place from a very young age when children begin to discover and get familiar with the environment they live in (Mason, 2008). For instance, when children observe and learn that traffic lights show green followed by amber then red in this order repeatedly, they could articulate that traffic lights have a continuous repeating pattern. Besides learning to generalise through experience, generalisation also occurs in other different learning situations, for instance, in the learning of multiplication in school. When children detect that 3×2 and 2×3 have the same product and so do 5×2 and 2×5 as well as 6×3 and 3×6 , they might be able to express the idea that the order in which two numbers are multiplied is not important. According to Mason, children are said to have made a mathematical generalisation because they can communicate what they have seen in just a few examples to all cases.

Besides arithmetic, generalisation also filters through many other areas in the context of mathematics learning. It can appear in the learning of algebra and geometry, where laws and theorems can be deemed as generalisations (Mason, 1996). A situation for algebra is the commutative law for multiplication. In the context of geometry, when pupils are given triangles of various shapes and sizes and are tasked to examine the sum of the interior angles of a triangle, they may conclude that the sum is always 180° regardless of the types (i.e., equilateral, isosceles or scalene) and sizes of triangles. So, this activity, although based on just a few specific triangles, may already be adequate to lead them to make some form of generalisation about the sum of the interior angles of any triangle being 180° . Generalisation is thus regarded as a big idea in mathematics, with many researchers and educators going as far as to affirm that it is at the heart of mathematics (Kaput, 1999; Mason, 1996). This is because generalising is a major and significant skill essential for mathematics learning.

Researchers have studied the approach pupils use to create expressions of generality for patterns for many years and recorded a few kinds of pattern generalisations (Radford, 2008; Rivera & Becker, 2005). Radford differentiated between arithmetic and algebraic pattern generalisations. An arithmetic pattern generalisation emerges when pupils engage in empirical counting to recognise the commonality and generalise it to the rest of the terms without presenting a general rule for finding any particular term. It often occurs when pupils use the common recursive strategy to continue the pattern by using the common difference between successive terms. Although recognising the recursive relationship between successive terms assists to extend the pattern, it does not, however, predict any term instantly when given its position in the pattern. In contrast, an algebraic pattern generalisation rests on “the capability of grasping a commonality noticed on some particulars (say $P_1, P_2, P_3, \dots, P_k$); extending or

generalising this commonality to all subsequent terms (P_{k+1} , P_{k+2} , P_{k+3} , ...), and being able to use the commonality to provide a direct expression of any term of the sequence” (Radford, 2008, p. 84). In other words, the observed pattern structure is abstracted, or symbolised in Radford’s language, into an expressed generality. By involving in structure recognition, the algebraic generalisation is therefore believed to be more complex than the arithmetic type, which fails to induce the “algebraic nature” (Radford, 2008, p. 85) of the former type despite establishing a generalisation.

There appears to be differing perspectives to consider when searching for a sufficient and workable definition for the term generalisation. In this paper, generalisation is defined as a “process which engages learners in at least one of the following three activities: (1) identifying a commonality across cases, (2) extending one’s reasoning beyond the range in which it originated, or (3) deriving broader results from particular cases” (Ellis, 2007, p. 197). According to Ellis, generalisation is a dynamic rather than static process. Ellis’ definition of generalisation is espoused in this paper because of its applicability to generalising tasks involving number and figural patterns (see Figure 1). Thus, to be able to organise information meaningfully in the generalising tasks involving number and figural patterns, the human mind detects what remains the same and what is changing and constructs an appropriate rule. According to Mason (1996), this reduces the demands on human concentration so that the mind can function efficiently.

Developing Algebraic Thinking using Problem-Solving Approach

Bednarz and Janvier’s (1996) model differentiates problems that require arithmetic thinking from those that demand algebraic thinking. Problems where the data is ‘connected’ involve arithmetic thinking to solve. Pupils use the provided known data to solve the unknown value. However, pupils generally require algebraic thinking when problems have ‘disconnected’ situations. In such problems, links between the data could not be determined easily. The following is an example that illustrates an arithmetic type problem: ‘Jane baked 44 cookies on Tuesday. She baked 17 more cookies on Tuesday than on Monday. How many cookies did Jane bake altogether?’. Here the conditions are clearly connected, and pupils can use the ‘knowns’ to solve for the unknown. The next example illustrates an algebraic type problem: ‘Printers A, B and C can print 9600 pieces of paper in an hour altogether. Printers A and B can print the same number of pieces of paper in an hour. Printer C can print 1500 pieces of paper more than Printer A in an hour. How many pieces of paper can Printer C print in an hour?’. Here the conditions are disconnected, and pupils must work with these different conditions to solve for the unknown. In the problem-solving approach to algebraic thinking, the equation is the problem-solving tool and the ability to formulate appropriate equations is of utmost importance. (Bednarz & Janvier, 1996). However, in the Singapore mathematics curriculum, the formulating of algebraic equations using letters as unknowns to solve a given word problem is introduced only at the first year of secondary school.

Developing algebraic thinking using a problem-solving approach can build upon and extend the teaching practices used within many classrooms. However, it may drive some teachers to view problem solving from a different perspective. At a minimum it suggests seeing problem solving as an opportunity to enrich and promote pupils’ thinking rather than the ‘ferreting out’ of an answer. It is evident that a problem-solving approach evolves and builds upon a pupil’s ability to read, explore and think about the mathematical concepts within a problem. Mathematical problem solving is the central focus of the pentagonal framework that

encapsulates the Singapore mathematics curriculum. In fact, the Singapore primary mathematics document states that “basic algebraic thinking is embedded in problem-solving processes” (Ministry of Education, 2023, p.45).

The goal of the mathematical problem-solving approach at the primary level in Singapore is to offer pupils with a tool that could help them in solving non-routine word problems and the model method provides pupils without recourse to equations, access to a less abstract method to solve algebraic word problems (Kho, 1987). In Kho’s paper ‘Mathematical Models for Solving Arithmetic Problems’ presented at the 1987 ICMI-SEAMS, he described the various types of models and highlighted the efficacy of the model method. He stated that:

Without using the model, one may have to resort to the algebraic method to solve structurally complex problems. However, the ‘model’ method is less abstract than the algebraic method and can be introduced to pupils before they learn to solve algebraic equations. Indeed, the models serve as good pictorial representations of algebraic equations. (p.349)

Teaching primary pupils the model method is part of the Singapore mathematics curriculum. Primary school pupils are also taught to analyse the problem situation and relationship when they solve word problems using the model method. This provides an opportunity for pupils to think algebraically. The model drawing is a structure comprising rectangles and numerical values which embody all the information and relationships shown in a given word problem. The rectangles replace the unknown represented by letters in equations and the rectangle, known as a unit, becomes the “generator” (Bednarz & Janvier, 1996) of the model about which other relations in a given problem is constructed. Identifying and articulating the parts that together form a whole is an algebraic thinking process when solving word problem. Using a problem-solving approach to develop algebraic thinking and providing an algebraic perspective of mathematics may enhance the long-term learning trajectory of many primary pupils.

Sample Mathematical Tasks that Promote Algebraic Thinking

Generalisation and problem-solving approaches seek to promote algebraic thinking. Through careful design of mathematical tasks, primary mathematics teachers can promote algebraic thinking in their mathematics lessons. In my work with primary mathematics teachers in Singapore, I frequently hear the concerns about infusing algebraic thinking skills in mathematical problems. Amongst others, primary mathematics teachers also wish that their pupils develop the following algebraic thinking while solving mathematical tasks: Generalising and specialising as well as analysing parts and whole. The following section describes two mathematical tasks that could promote algebraic thinking at the primary level.

Generalising and Specialising

When a problem solver is faced with a generalising task, a robust approach is to investigate its pattern by examining particular cases to arrive at a recursive rule. Such specialisation is the commencement to an inductive approach to learning and is considered a natural process for children’s learning (Burton, 1984). Burton further stated that each case provides an opportunity for handling components that are concrete in the child’s thinking, whether they are physical displays or ideas. Nonetheless, generalising involves moving from a few cases to making statements about a wider collection of cases or examining one case in detail to make sense of interrelationships between instances more generally (Jeannotte & Kieran, 2017; Wood et al.,

2006). According to Burton, the generalisation statements or predictive rules occur to be the building blocks used by pupils to create order and meaning out of a vast quantity of information. There is a direct link between the processes listed by Burton and the thinking skills listed in the Singapore primary mathematics curriculum — identifying patterns and relationships where pupils “recognise the specific variations between two or more attributes in a relationship that yields a reliable or repeated scheme” (Ministry of Education, 2000, p. 131).

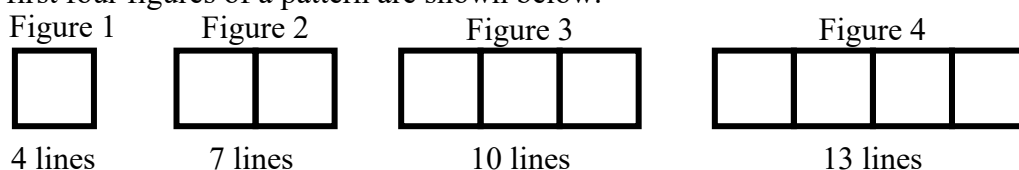
A common approach to generalisation activities in school algebra is to set them in the context of figurate patterns (for example: a row of touching squares formed by matchsticks, with a relationship between the numbers of squares and matchsticks; or a row of touching tables surrounded by chairs, with a relationship between the numbers of tables and chairs). To help pupils to succeed, teachers might begin a presentation around a generalising task by spending time exploring and discussing the pattern seen in the task. This approach allows pupils to observe and explain numerical relationships in terms of the figurative pattern’s structure. The process of generalising and specialising is illustrated in mathematical task 1 (Figure 1) which is a generalising task.

In order for the pupils to examine patterns and note regularities in a set of values and figures, mathematical task 1 involves examining patterns from the number of squares and lines. Mathematical task 1 depicted only one configuration and was equivalent to asking pupils to find the figure number of the given term (i.e., figure 4 has exactly 13 lines. Which figure will have exactly 124 lines?). There are two critical questions which are helpful tools to help pupils identify the patterns and relationships as well as to discover the recursive rule and predictive rule. The two questions are: ‘what remains the same?’ and ‘what changes?’. This question, ‘what remains the same?’, directs pupils to look for sameness in the generalising task. This question, ‘what changes?’, directs pupils to look for changes within the same generalising task. At the first glance from the table and figures in mathematical task 1, the number of lines is increased by 3 based on the relationship between the previous figure and the next figure. The recursive rule is adding three lines to get the next figure. Anecdotal evidence provided by teachers and primary pupils indicate that coming up with the predictive rule is a much higher level of abstraction than that required for the construction of the recursive rule. Similarly, as highlighted by Warren and Cooper (2008), such a task can be difficult for many pupils if they cannot make a link between the number of squares, figure number and the number of lines. Prompting questions should be used to direct pupils’ attention at the generalising task so that they can abstract the relationship between the figure number and the number of lines. Prompting questions such as ‘how do you use the same rule to make figure 2 and figure 3?’ and ‘how do you use the same rule to make figure 3 and figure 4?’ would help pupils to understand that they must use the same recursive rule repeatedly to construct each subsequent figure. Pupils could be encouraged to articulate what they see in the pattern for mathematical task 1 and then asked to extend the pattern by predicting some terms that are both near to and far from figure 16 in the pattern. This will lead the pupils to construct an initial predictive rule for the total number of lines for figure 4. The computation of these terms is a remarkably helpful exercise for those who do not immediately recognise any pattern structure.

Mathematical task 1 shows how pupils’ explanations can be tested by asking pupils to perform a far prediction of what the next result or predictive rule would be. The aim is to have pupils see a pattern that they can generalise into a predictive rule. As pupils become more proficient with verbalising a predictive rule, the teacher could show them how the rule can be written

Mathematical Task 1 (Primary 6) - Generalising and Specialising

The first four figures of a pattern are shown below.



(a) The table shows the number of squares and lines used for each figure and then complete the table below

Figure	1	2	3	4	5	6	7	10	15
Number of squares	1	2	3	4			7		
Number of lines	4	7	10	13	16	19			

(b) Which figure will have exactly 124 lines?

Solution

(a) For the number of lines, 4, 7, 10, 13, 16 and 19, the difference between the consecutive numbers is the same and it is plus 3.

Thus, the recursive rule is +3

Figure	1	2	3	4	5	6	7	10	15
Number of squares	1	2	3	4			7		
Number of lines	4	7	10	13	16	19			

(b) To arrive at the Predictive Rules

1. Total number of lines in Figure 1 = 1 line + 3 lines
2. Total number of lines in Figure 2 = 1 line + 2 groups of 3 lines or 2×3
3. Total number of lines in Figure 3 = 1 line + 3 groups of 3 lines or 3×3
4. Total number of lines in Figure 4 = 1 line + 4 groups of 3 lines or 4×3

Near Prediction

Total number of lines in Figure 7 = 1 line + 7 groups of 3 lines or $7 \times 3 (= 21)$

(Check your answer by applying the recursive rule)

Far Prediction

Total number of lines in Figure 16 = 1 line + 16 groups of 3 lines or $16 \times 3 (= 48)$

Use words to describe the Predictive Rule

Number of lines is equal to One added to Figure Number multiplied by 3 or

Number of lines = $1 + \text{Figure Number} \times 3$

Therefore, $124 = 1 + \text{Figure Number} \times 3$

Figure Number = $(124 - 1) \div 3$

= 41

Figure 1. A generalising task that develops algebraic thinking

words to describe the predictive rule. This is because primary school pupils may find it a challenge to articulate or describe the predictive rule. This has certainly created an opportunity for pupils to explore, appreciate figurate patterns, and 'see' the relationship in terms of number of lines and figure number. From a pedagogical viewpoint, computing near and far terms is beneficial as it helps pupils who could articulate a recursive relationship realise two important understandings: (1) determining a near term requires them to know the term immediately

preceding it and the differences between consecutive terms, and (2) the recursive approach is not an adequate method for determining a far term because the term immediately preceding it may not be available. Essentially, central to all these computations is to let pupils recognise that there is a need to devise a general rule from both the recursive and predictive rule.

Analysing Parts and Whole

In school, algebraic problems are often presented as stories or word problems (see Figure 2 for an example). Like pupils elsewhere, upper primary school pupils in Singapore often find these algebraic word problems tough. To give them better access, upper primary school pupils are taught the model method. This process emphasises the identification of the part or unknown that is the ‘generator’ (Bednarz & Janvier, 1996) of relationships in a given problem. Identifying the generator and abstracting and articulating the procedural relationships between the parts as well as analysing parts and whole are key to problem solving via the model method. Analysing parts and whole in a mathematical task is to recognise and articulate the parts that together constitute a whole (Ministry of Education, 2000). Additionally, the problem solver should be able to (1) state the parts, (2) determine the value, function and/or purpose of each part, (3) identify the relationship between or among the parts, (4) identify how the parts contribute to the whole and (5) draw conclusions from the relationship between the parts and the whole.

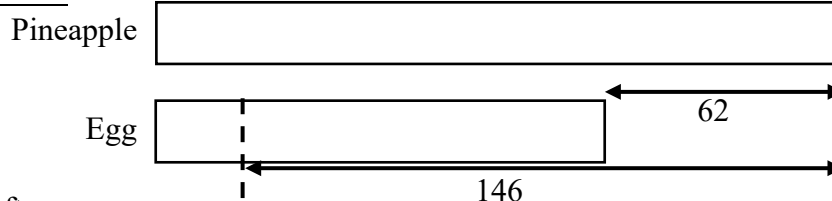
Mathematical task 2 in Figure 2 can be used to provide pupils opportunities to identify and explain how each part of the computation contributes to the whole. In this case, the whole refers to the number of tarts to be sold. This entire process of algebraic reasoning involves the flexibility in thinking about the how the number of pineapple tarts and egg tarts are related to the whole (refers to number of tarts to be sold). Pupils are taught to draw models to represent relationships presented in word problems (see Figure 2). Perhaps primary school teachers, when teaching pupils the model method, may need to be more explicit that the rectangles are used to represent the unknowns as presented in the algebraic word problems. To solve mathematical task 2, two sets of model drawings, before the change and after the change, are needed. The ‘before’ model represents the tarts situation before any pineapple tart was sold while the ‘after’ model represents the changed situation. The ‘before’ model shows that at first there are 62 more pineapple tarts than egg tarts. In the ‘after’ model, a shorter rectangle is used to represent the number of pineapple tarts as the total number of pineapple tarts has decreased. The grey rectangle represents 146 pineapple tarts that are sold. The ‘after’ model shows that the number of egg tarts is 5 times as many as the number of pineapple tarts after 146 pineapple tarts are sold. Pupils are expected to solve for the unknowns by analysing the quantitative relationships between the rectangles. In each case the rectangles allow pupils to treat the unknowns as if they are known and here each rectangle is a unit which represents more than one tart. Thus, the generator or unit can be found by comparing the number of egg tarts (5 units) in the ‘before’ drawing with that found in the ‘after’ drawing (1 unit). From the model drawings, pupils see that four units of egg tarts in the ‘after’ drawing represent $146 - 62 = 84$ egg tarts. Therefore, one unit of the egg tarts represent $84 \div 4 = 21$ egg tarts. Finally, since the total number of egg tarts is represented by 5 units, the number of egg tarts is $5 \times 21 = 105$ at first. Figure 2 illustrated that when the model method is used to solve algebraic word problems, the information captured by the model is translated into arithmetic expressions, which can be evaluated and a single numerical answer results from operating on the arithmetic.

Mathematical Task 2 (Primary 5) - Analysing Parts and Whole

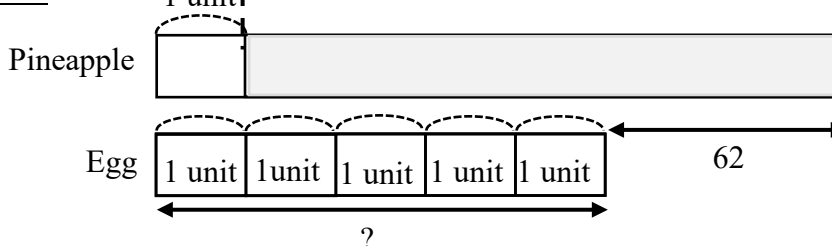
Ann made some tarts to sell. There were 62 more pineapple tarts than egg tarts.
After 146 pineapple tarts were sold, the number of egg tarts was 5 times as many as the number of pineapple tarts.
How many egg tarts were there at first?

Solution

Before



After



Unitary Method

$$4 \text{ units} = 146 - 62 = 84$$

$$1 \text{ unit} = 84 \div 4 = 21$$

$$5 \text{ units} = 5 \times 21$$

$$= 105$$

Figure 2. An algebraic word problem that develops algebraic thinking

Implications

The generalising task and algebraic word problem in Figure 1 and Figure 2 exemplify how teachers could use questionings to develop primary school pupils' algebraic thinking. The generalising task and algebraic word problem that were discussed highlighted the different learning experiences that pupils will gain when they work on different types of problems. This is only possible when the generalising tasks and algebraic word problems that teachers use in their classrooms go beyond computations and rote algorithms. The generalising and problem-solving approach are just first steps towards making the use of generalising tasks and algebraic word problems in the classroom a meaningful one where emphasis is on the process (algebraic reasoning and thinking) rather than the product (final answer). In addition, the ongoing use of generalising tasks and algebraic word problems will provide opportunities and possibilities for pupils to develop their algebraic thinking.

Conclusions

Algebraic thinking is important in any mathematics curriculum. The Singapore primary mathematics curriculum provides extensive and eclectic resources for developing algebraic thinking through generalisation approach and problem-solving approach. Offering overt opportunities for generalising and specialising as well as analysing parts and whole enables the solver to reflect on his or her own thinking more deeply. If pupils and teachers routinely spent the first six years of primary education simultaneously developing arithmetic and algebraic thinking, arithmetic and algebra would come to be viewed as intimately connected.

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