

Mathematical Thinking: Task Design and Implementation in Primary Mathematics Classrooms

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Mathematical thinking is an important learning goal across the world. This chapter will discuss how mathematical thinking can be integrated into primary mathematics classrooms by examining some mathematical tasks, and teaching through problem solving. Considerations for task design and implementation to support students on their path to thinking about mathematics meaningfully and with flexibility will also be elucidated from the discussion.

Keywords: Mathematical Thinking, Task Design, Primary Mathematics

Introduction

The Fourth Industrial Revolution is characterised by dramatic technological advancement. This technological advancement has resulted in the redesigning of work in many industries but at the same time it has also created new jobs (Lee et al., 2021). The 2022 Skills Demand for the Future Economy report by SkillsFuture, Singapore (SkillsFuture Singapore, 2022), has identified several skills such as problem solving, thinking, decision making, among other critical core skills for the future economy, as important for business transformation. In an article by a local newspaper, The Straits Times, Abdullah (2023) reported that the Education Minister of Singapore, “likens ChatGPT to a calculator for learning mathematics, but said that ChatGPT can be a useful tool for learning only when students have mastered basic concepts and thinking skills.” The emphasis on thinking skills with the advent of technological/digital tools and artificial intelligence is critical to prevent the erosion of thinking skills and skills degeneration. “First and foremost, it [education] should teach those young people to think” (Pólya, 1963, p. 605). To prepare our students for the future economy, it is important to revisit this topic of mathematical thinking and reflect on how we can continue to develop children’s mathematical thinking and reasoning so that they are not only consumers of mathematics but also, creators of mathematics, to innovate and work creatively. For example, the discovery of non-Euclidean geometry paved way for Albert Einstein’s theory of general relativity and shed light to planetary orbits that has practical uses in GPS navigation (BBC News, 2019).

According to Bishop et al. (2022, p. 10), “Pedagogies that substantively incorporate students’ mathematical thinking are associated with” improved student learning, productive mathematics dispositions and identities, student engagement in mathematical practices and more equitable classroom environments. However, teaching to cultivate mathematical thinking does not tend to happen in some mathematics classrooms. This could be attributed to (i) people’s perception of mathematics e.g. one’s perception of mathematics as a static, structured system of facts,

procedures, and concepts versus perception of mathematics as an exploratory, dynamic evolving discipline affects one's understanding of mathematical thinking; (ii) the challenge in developing classroom environments that is highly responsive to students' mathematical thinking; and (iii) teachers' understanding of mathematical thinking (Isoda and Katagiri, 2012).

What Does it Mean to Think Mathematically?

"In Aesop's fable, a thirsty crow spied a pitcher containing a small amount of water. Unfortunately, the water was out of reach of the crow's bill, but next to the pitcher was a pile of stones. The crow began placing the stones in the pitcher, thereby raising the water until it could drink" (Emery and Clayton, 2004, p. 1903). Was the crow thinking mathematically? What does it mean to think mathematically? Mathematical thinking is a broad term containing many perspectives and meanings. Each researcher dealing with mathematics education has his or her own perspective on this topic. Isoda and Katagiri (2012) define mathematical thinking as "an ability to think and make decisions" (p. 32). They emphasize the importance on how to learn by oneself, and attaining the abilities required for independent learning. Another group of researchers conceptualize mathematical thinking as the activity of using mathematical concepts mathematically. According to Kooloos et al. (2022), mathematical thinking takes on different forms such as conceptual understanding, problem solving, ways to work with mathematics, e.g. generalizing and specializing, or modelling. Mathematical thinking can indeed mean all those activities and they could be inherently intertwined, or "as an amalgam of overlapping and mutually interdependent activities" (Kooloos et al., 2022, p. 505). Many types of mathematical thinking processes are explored and identified in mathematics education e.g. web of mathematical thinking is one attempt (Karadag, 2009). There are efforts to categorize these cognitive activities, but it is a very challenging endeavour because each of these activities is closely connected and interwoven with others.

Singapore School Mathematics Curriculum and Mathematical Thinking

According to Stupel and Ben-Chaim (2017, p. 86), "One goal of the mathematical curricular standards in many countries is to develop students' ability to think, prove, and reason (Lowrie et al., 2012; NCTM, 2000)." Thinking skills are emphasised in the Singapore School Mathematics Curriculum (MOE, 2000, 2006, 2012) and are part of the mathematical processes involved in the process of acquiring and applying mathematical knowledge. According to the Singapore School Mathematics Curriculum document in 2006, "students should use various thinking skills and heuristics to help them solve mathematical problems. Thinking skills are skills that can be used in a thinking process" (MOE, 2006, p. 8). Examples of thinking skills listed are classifying, comparing, sequencing, analysing parts and wholes, identifying patterns and relationships, induction, deduction and spatial visualization (MOE, 2006). In the 2020 Singapore School Mathematics Curriculum document, "abstraction is at the core of mathematical thinking" (MOE, 2020, p. 14). "Purely cognitive views see abstraction as ascending from concrete to abstract" (Ozmantar and Roper, 2004, p. 481). For example, "the extraction of what is common to a number of different situations" (Dienes, 1963, p. 57). An operational definition of abstraction was defined by Hershkowitz et al.'s (2001) model as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" (p. 195).

The Singapore School Mathematics Curriculum has over the years consistently emphasize that thinking skills should be integrated in mathematics teaching and learning with mathematical

problem solving being the focus. The development of concepts, skills and its underlying processes “together with the explication of thinking skills and the integration of IT in mathematics teaching and learning will give leverage to the development of problem solving” (MOE, 2000, p. 13). “It is important also to incorporate the use of thinking skills and heuristics in the process of developing skill proficiencies” (MOE, 2006, p. 7). “The teaching of process skills should be deliberate and yet integrated with the learning of concepts and skills” (MOE, 2012, p. 31). The ways of thinking and approaching a problem needs to be introduced to students since the central focus of the mathematics curriculum is the development of mathematical problem solving skills (MOE, 2020). Opportunities to cultivate mathematical thinking are abundant in primary mathematics instructional materials. Mathematical thinking is used when solving problems “in any context, although questions of a mathematical nature might more readily expose such thinking” just like scientific method which does not necessarily apply solely to science (Burton, 1984, p. 38). In the next few sections, we will examine a few mathematical tasks to elucidate mathematical thinking that can be developed through the tasks.

Whole Number Subtraction Algorithm

The subtraction algorithm involving regrouping can be a challenge for students in the primary grades. The difficulty is magnified when students learn to perform the algorithm with instrumental understanding and forget the steps which are meaningless to them. Subtracting a whole number from 100 is a common learning difficulty for some primary school students (e.g. subtract 19 from 100). The number of regroupings needed in the standard algorithm can pose a challenge to some students (See Figure 1b). When learning to deal with this type of subtraction problem, key thinking skills such as analysing parts and whole can be made visible to facilitate student’s relational understanding of the subtraction algorithm. The widely used Hindu-Arabic numeration system is characterised by positional base ten principle, hence, 100 can be decomposed into 9 tens and 10 ones. This ‘analysing parts and whole’ thinking can be illustrated using base ten blocks (See Figure 1a), a manipulative that can support students’ modelling of abstract numeral concepts. In the physical base-ten blocks manipulative, a single cube represents the basic unit one, 10 cubes make up a tens rod and 100 cubes make up a hundred flat. The teacher can use the base-ten blocks to explain the isomorphism that emphasises discrete quantities of 100 before demonstrating the decomposition of the hundreds flat into 9 tens rod and 10 single cubes (See Figure 1a). This decomposition can then be recorded in the subtraction algorithm (See Figure 1b).

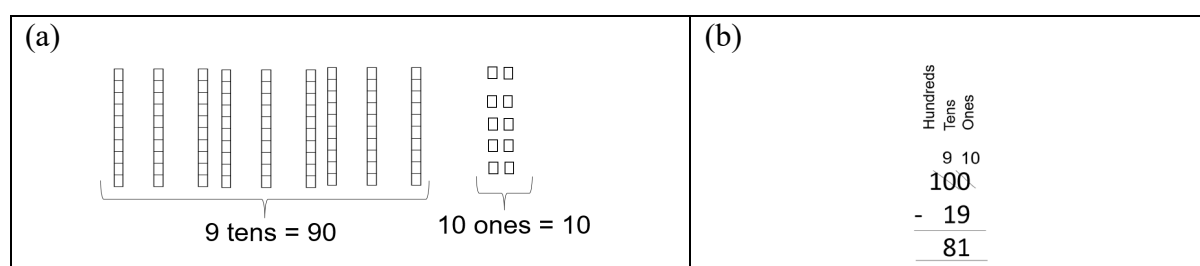


Figure 1. Decomposition of 100 into 9 tens and 10 ones

From Figure1(a), subtract 19 from 100 becomes subtract 9 from 10 ones followed by subtract 1 ten from 9 tens resulting in 8 tens and 1 one. This can be recorded correspondingly in the subtraction algorithm in the tens and ones place (Figure 1b). Students can also compute $100 - 19$ flexibly without limiting themselves to the subtraction algorithm once they are able to decompose 100 into its various parts using the thinking skill of ‘analysing parts and whole’.

For example, they can also decompose 100 into 80 and 20. $100 - 19$ becomes $80 + 20 - 19$ and mental calculation gives the answer $80 + 1 = 81$.

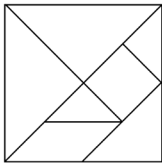
In this example, we see that base ten blocks can be an effective tool for teachers to translate abstractions into a form that empower learners to connect new knowledge to students' pre-requisite knowledge. "Because students' abstract thinking is closely anchored in their concrete perceptions of the world (Thompson, 1992), actively manipulating these materials allows learners to develop a repertoire of images that can be used in the mental manipulation of abstract concepts." (Moyer, 2002, p. 176). Furthermore, when students are guided to translate between representations, they are enabled to extend their learning when they deal with this type of subtraction problem involving larger numbers. For example, when they encounter situations such as $1000 - 19 = ?$, they can apply the thinking skill of 'analysing parts and whole' to approach the task.

Fractions

Fractions are very important from the mathematics education perspective. Basic understanding of fractions is needed in daily life and fractions are an essential building block in the number strand. However, learning fractions has been reported to be difficult across countries and this difficulty persists in spite of the many years of instruction on fractions that children received (Tunç-Pekkan, 2015). Understanding fractions requires a higher level of abstraction than understanding natural numbers (DeWolf et al., 2016; Empson et al., 2011). There are many potential sources of difficulties with fractions (Lortie-Forgues et al., 2015). Reading and interpreting fractions itself pose a challenge to some students as it requires the ability to think about the whole and the parts simultaneously. There are many activities in instructional materials that aim to develop these two mental actions, one of which is tangram activities. The tangram contains seven geometric shapes: 2 small triangles, 1 medium triangle, 2 large triangles, 1 square and 1 parallelogram, that can form different patterns. The tangram pieces can be used to develop the two mental actions. It can also be used to provide many other mathematical learning experiences that develop mathematical thinking. Figure 2 shows some examples of tangram activities.

In the tangram activities shown in Figure 2, students can move and rotate the tangram pieces, compare, analyse each piece of the tangram with the other pieces before identifying any patterns and relationships. Task (d) is the reverse of tasks (a), (b) and (c), which encourages flexibility in thinking as there are different ways to compose a figure to represent $\frac{5}{16}$ of the big square. Krutetskii (1969) characterizes flexible thinking as reversibility of thought — "the ability to establish two-way relationships that comes from the ability to make the transition from a direct association to its corresponding association" (Lee, et al., 2020, p. 3). Cognitive flexibility is defined as the ability to use existing knowledge flexibly, arrange and respond the knowledge appropriately to the changing situations (Spiro & Jehng, 1990). Cognitive flexibility is reported to be an important factor to be successful in mathematics (Baroody, 2003; Star & Newton, 2009). In mathematics education, flexibility has been increasingly valued to prepare students for the rapidly changing society. Geometric thinking and spatial reasoning can also be developed from the tangram activities. The tangram activities can also be extended to other mathematical topics such as area, geometry, and ratio, e.g. area of the two small triangles is the same as the area of the square.

Describe the shapes that you see in the tangram puzzle.



(a) What fractional part of the whole puzzle is a large triangle?
 (b) What fractional part of a large triangle is a medium triangle?
 (c) Write each tangram piece as a fraction of the big square.
 (d) Which are the three tangram pieces that can be used to make a figure that represents $\frac{5}{16}$ of the big square?

Figure 2. Examples of tangram activities

Area of Rectangle

There are many rules that students need to learn and remember in mathematics. Remembering the different rules and applying the rules correctly may be challenging for some students. One common learning difficulty students face in area measurement is the formulas for area concept. Figure 3(b) shows an example of a common error in area measurement due to an incorrect formula being applied. The error in Figure 3(a) also reveals students' difficulties in understanding the difference between the concept of area and perimeter.

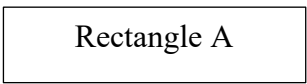
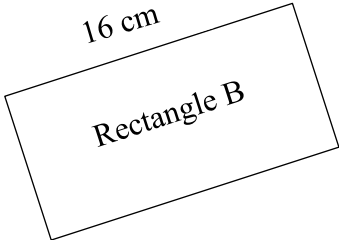
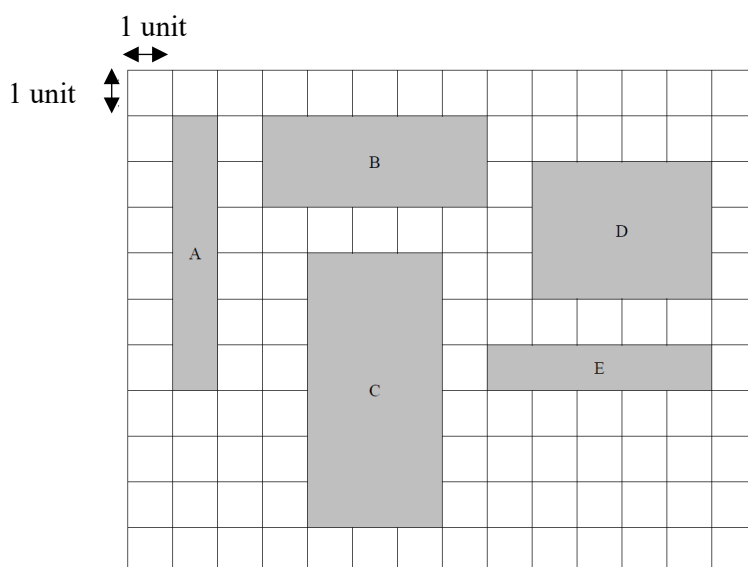
(a)  Area of rectangle A = 30 cm²	(b)  Area of rectangle B = 24 cm²
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Figure 3. Examples of errors in finding area of rectangle

For information to be stored permanently and retrieved, it must be semantically encoded into long-term memory. Information in the long-term store remains for most of a person's life (Tan et al., 2017) while "information that remains in the sensory register or the short-term store deteriorates according to each store's decay rate" (Stout and Klett, 2020, p. 119). Meaningful content and connections to other known information can help improve long term memory (Edge, 2009). For the mathematics to be meaningful, students can be engaged in activities to explore, construct meanings, and uncover abstract mathematical concepts or results. For example, an activity worksheet can be designed to guide students to collect data for the length and breadth of a given set of rectangles of varying dimensions (see Figure 4). The given set of rectangles can be drawn on square grids to enhance students' understanding about the area attribute. Students can record the length and breadth of the given set of rectangles in the activity worksheet, re-examine the data, look for patterns and relationships, predict, conjecture, confirm and discover the rules for area of rectangle. In this activity, students can engage in inductive thinking to discover and gain relational understanding (Skemp, 1976) of the concept of area of rectangle. According to Isoda and Katagiri (2012), inductive thinking is a method of thinking that involves "attempting to gather a certain amount of data; working to discover rules or

properties common to these data; confirming the correctness of the inferred generality with new data” (p. 53). Induction refers to “drawing a general conclusion from clues gathered (from specific to general)” (MOE, 2000, p. 51).

Count the number of unit squares to find the area and perimeter of each rectangle.



Complete the table.

Rectangle	Length (unit)	Breadth (unit)	Perimeter (unit)	Area (unit ²)
A				
B				
C				
D				
E				

What patterns and relationships do you notice in the table?

Figure 4. Area and perimeter of rectangle adapted from Yeo (2021, p.133)

Problem Solving Heuristics

“Thinking skills and heuristics are essential for mathematical problem solving” (MOE, 2012, p. 17). Heuristics is often known as mathematical problem solving techniques or approaches in the solution process (Goldenberg et al., 2003; Pólya, 1945/1973; Schoenfeld, 1985). Heuristics is described by Foong (2009) as general strategies through which a problem solver moves towards a solution. It is different from algorithms and involves a decision-making process (Kaitera and Harmoinen, 2022). Thinking skills and heuristics can be used to help students solve different types of mathematical problems. The 2006 Singapore School Mathematics Curriculum document (MOE, 2006) listed several heuristics grouped into four categories. Some of the heuristics include draw a diagram, make a list, look for patterns, make suppositions

etc. Next, we examine the following real-world mathematics problem to unpack possible thinking skills that can be developed through the problem: During a fund-raising event, 50 pencil cases and water bottles were sold for \$719. Pencil cases were sold for \$13 each and water bottles were sold for \$16 each. Find the number of water bottles sold during the event.

There are several ways to approach the above heuristic task. Table 1 shows a possible solution method using making a list to generate a set of data and the data is recorded in a table. From the first row of data generated, students can compare the cost of each water bottle and pencil case and identify its relationship with the total cost of 25 pencil cases and 25 water bottles. The data in the first row shows that the total cost \$725 exceeds the given amount of \$719. Students can then deduce that they can purchase fewer water bottles to decrease the total amount spent since a pencil case cost less than a water bottle. Students then continue to systematically increase or decrease the number of water bottles and pencil cases to generate more data. By identifying relationships and patterns from the data represented in the list, students can thereby deduce the solution.

Table 1.

One possible solution method

Number of pencil cases	Number of water bottles	Total Cost	
25	25	$25 \times \$13 = \325 $25 \times \$16 = \400 $\$325 + \$400 = \$725$	\$725 (incorrect)
26	24	$26 \times \$13 = \338 $24 \times \$16 = \384 $\$338 + \$384 = \$722$	\$722 (incorrect)
27	23	$27 \times \$13 = \351 $23 \times \$16 = \368	\$719 (correct)

When the student “uses what is already known as a basis and attempts to explain the correctness of a proposition in order to assert that something can always be stated” the student is engaged in deductive thinking (Isoda and Katagiri, 2012, p. 59). Deduction also refers to “inferring various specific situations or examples from given generalisations (from general to specific)” (MOE, 2000, p. 51). According to Anghileri (1995, p. 19), human beings “appear to be very able to engage in the process of induction”, however, “relatively less well-equipped” when it comes to deductive reasoning and thinking. The heuristic task discussed in this section offer opportunities to develop thinking skills, and in particular, deductive thinking.

As shown from the examples above, there are many opportunities to develop mathematical thinking in our mathematics lesson through existing instructional materials. Mathematical thinking can also be deliberately incorporated and integrated when designing mathematical tasks for meaningful implementation. In order for mathematical thinking to be explicitly integrated into the mathematics lessons as an instructional goal, we can ask ourselves the following questions when designing our instructional materials: (1) What are the thinking skills that can be developed through the task? (2) What is the task’s potential to engage students with the five critical tools of mathematical thinking — reasoning, communication, connections, representations, problem solving (Katz, 2014). In other words, does the task have the potential

to engage the students to reason and communicate their thinking? Does the task have the potential to engage the students in making various connections and representations? For example, the heuristics task above is rich in opportunities to engage students in the five critical tools of mathematical thinking. Students can be encouraged to explain their solution strategies, the connections, and representations they make as they try to solve the problem.

Key to Recognizing and Using Mathematical Thinking?

“Mathematical thinking does not automatically emerge from a study of mathematics” (Burton, 1984, p. 44). An overly conscientious concentration on mathematical content may have the danger of obstructing the development of mathematical thinking. “The key to recognizing and using mathematical thinking lies in creating an atmosphere that builds confidence to question, challenge and reflect” (p.47). We can start building this atmosphere by inviting students to pose questions from pictures in their everyday life (e.g. see Figure 5). As they notice and wonder, they are encouraged to conjecture, challenge the hidden assumptions, and reflect on what they have uncovered (Burton, 1984). What students notice and wonder can be skilfully used to connect to the learning outcomes of the lesson. If we aim to cultivate children who would like to think by themselves, “this means that the child has his or own wish to explore mathematics” (Isoda and Katagiri, 2012, p. 23) and this mathematical attitude is one of the driving forces of mathematical thinking. When students are actively and creatively involved in knowledge construction, they see mathematics as a process and start to build a sense of ownership of the mathematics, pivotal in encouraging “reflection on the experiences of life and fascination for the relationships that are mathematics” (Anghileri, 2005, p. 9). Conscious reflection on what is being done and why it is being done is an important element in learning (Burton, 1984).



Figure 5. An example of a notice and wonder task

In inquiry-based classrooms students are expected to engage in proposing, defending, and responding to mathematical ideas, conjectures and arguments (Goos, 2004). When students ask and seek answers to their own questions during the inquiry process, learning is embedded (Laxman, 2013). “Students can be successful by learning to reason and think mathematically and apply ideas of mathematics” in an inquiry-based mathematics classroom (Katz, 2014, p. 8). The learning activities in inquiry-based learning can be easily identified in teaching through problem solving (Cheng, 2022; Yeo, 2021). According to Yeo (2021), teaching through problem solving can be enacted using four phases — *problem presentation* phase, *exploration* phase, *discussion of solutions and summary* phase, and *consolidation* phase. For the rest of this section, we discuss activities that can be carried out during the *discussion of solutions and*

summary phase to promote mathematical thinking using the lesson objective: To solve problems involving a whole number multiplied by a decimal number by using students' pre-requisites knowledge. The lesson will help students to also understand that multiplication can still be used when the multiplier is a decimal (Lerner, 2022). In the *problem presentation* phase, students can be presented with the problem: Bala bought 6 bottles of juice. Each bottle contained 0.8ℓ of juice. How much juice did Bala buy altogether? Students can work individually or in small groups to explore solutions to the problem by tapping upon their pre-requisite knowledge or intuitive knowledge. In the *discussion of solutions and summary* phase, time is used for whole class discussion of their solution attempts. In this stage, different solution methods were presented by the students to illustrate their different strategies. In the *consolidation* phase, questions can be posed to students to consolidate the learning activated by the initial problem.

Solution Methods and Comparisons across Solution Methods

Learning mathematics with multiple solutions has been discussed by a several authors (e.g., Schoenfeld, 1992; Stigler et al., 2000). Mathematical thinking can be fostered through multiple solutions (Cai & Kenney, 2000). Studies (e.g. Choy, 2020; Lerner, 2022) have reported possible solution strategies for a whole number multiplied by a decimal number. The following summarizes some of the solution strategies which depend on the pre-requisite knowledge and informal knowledge that students bring to the mathematics class:

- Repeated addition: Students use pre-requisite knowledge of multiplication as repeated addition $0.8 + 0.8 + 0.8 + 0.8 + 0.8 + 0.8$ to figure out the solution. It can be thought of as six groups of 0.8, “albeit with the equal size quantities as a non-whole number” (Lerner, 2022, p.51).
- Scaling: Students think of 0.8 as a whole number by multiplying it by 10 and then multiplying the final product by $1/10$. In other words, $6 \times 0.8 = (6 \times 8) \div 10$.
- Number line: Students use pre-requisite knowledge of the number line to figure out 6×0.8 .
- Decimals as fractions: Students rewrite decimal numbers as fraction equivalents $6 \times 0.8 = 6 \times \frac{8}{10} = 4\frac{4}{5}$.
- Estimation: Students derive from $6 \times 8 = 48$ before using estimation strategy to figure out the answer 4.8 because $6 \times 0.8 \approx 6 \times 1$.

The various possible solution strategies that students can generate using their pre-requisite knowledge and informal knowledge require students to replicate, restructure existing ideas of multiplication concepts in whole numbers, number line etc. in adaptive response to the decimal multiplication problem posed. The various solution methods also enrich students' problem solving repertoire and empower them to approach problems with flexibility as they can switch between the strategies. Providing opportunities for students to make comparisons across methods can stimulate students' thinking (Kaur et al., 2019). During the *discussion of solutions and summary* phase for the lesson on a whole number multiplied by a decimal number, mathematical thinking can be stimulated and advanced by comparing solution methods. The use of comparing solution methods is also known as comparing solution strategies (Rittle-Johnson and Star, 2007). When students explore and compare the various solutions for 6×0.8 from different perspectives and mathematical areas from their previous learning and intuitive knowledge, they are encouraged to recognise the similarities, differences and connections

across the solutions and reflect on their activities. They also deepen their understanding as they engage in reflection and sense making on the multiple solution methods, which in turn can promote the transfer of learning.

By considering multiple solution methods, students have the opportunity to make use of different representations. “Mathematics requires representations. In fact, because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas” (Kilpatrick et al., 2001, p. 94). A relationship between two quantities is an example of an abstract idea that can be represented using words, symbols, pictures, a graph etc. “Mathematical ideas are enhanced through multiple representations” (p. 95). Children demonstrate their understanding in three stages of representation: enactive, iconic and symbolic stage (Bruner, 1960, 1986). The enactive stage suggests the role of physical objects (Moyer, 2002). The Multi-Modal Strategy (Wong, 2015) is one approach to multiple representations comprising of six modes of external representation — concrete material, diagram, number (calculate), word, symbol and notation, and story. For the example 6×0.8 , activities that embody 6×0.8 can be used to construct understanding of multiplication concept, diagrams such as number line, rectangular array model can be drawn to represent 6×0.8 , the value of 6×0.8 is calculated, decimal notation, read 6×0.8 as multiply 6 by zero-point-eight and a story to represent 6×0.8 . Students can use representations in a number of ways — as tools for thinking (Bass, 2011), as problem solving skills, to explain their thinking to others, to reason, construct, invent, communicate about the mathematics, to support mathematical explanation (Selling, 2016). Studies have shown the value of learning mathematics with multiple representation (Brenner et al., 1997; White & Pea, 2011). Learning progresses through four stages: “using one representation, using multiple representations in parallel, making connections between parallel representations, and finally integrating [and coordinating] representations and moving flexibly between them (Dreyfus, 1991)” cited in Selling (2016, p.192). By comparing and connecting the various representations (for example, 6×0.8), students are provided the opportunity to analyse, evaluate, reflect, organise and decide how the representations are “part of an internal network” — “Just like how mathematics is understood by some experts as a complex system of relationships” (Cheng, 2022, p. 18). Students need to be provided with representatives in order to enhance their mathematical thinking and teachers play a critical role in designing such mathematical environments (Liggett, 2017). The multiple solution methods presented above for 6×0.8 also demonstrate the connection between different areas of mathematics (whole number multiplication, rounding numbers estimation etc.). Mathematics is an interconnected discipline and not just a collection of isolated topics hence making connections between mathematical ideas, facts, procedures, and relationships is necessary for mathematical understanding (Hiebert & Carpenter, 1992). When students make connections among mathematics ideas and between mathematics, they are enabled to think critically as they adopt different perspectives.

Successful implementation of the comparison of solution strategies in the classrooms can be challenging (Rüede et al., 2023). One of the challenges is “orchestrating classroom discussions in which solution strategies are compared” (Rüede et al., 2023, p.41). To overcome this challenge during the *discussion of solutions and summary phase*, solution strategies can be carefully selected and sequenced so that the reporting is cumulative” (Yeo, 2021, p. 129) to spiral mathematics ideas and thinking and at the same time connects the various strategies together.

Conclusion

“All children, irrespective of their ability or social background can enjoy the stimulation and fascination of mathematical thinking as well as developing self-reliance when they apply the knowledge they have acquired” (Anghileri, 1995, p. 9). To enhance students’ learning experience, mathematics teachers need to orchestrate opportunities for students to learn from mathematically meaningful tasks (Choy, 2020). Quality classroom discussions and interactions (e.g. teacher-student interactions, student-student interactions) can contribute to helping students develop and advance mathematical thinking and deep understanding of mathematics. Teachers can support students in this endeavour in several ways, such as, creating classroom norms that encourage students to participate by sharing their thinking and listening to others’ thinking (accountability to learning community), making statements that are as accurate as possible (accountability to accurate knowledge) and using evidence to support their statements in a logical manner (accountability to rigorous thinking) (Rüede et al., 2023). In order to foster and advance mathematical thinking and to orchestrate such opportunities for students, mathematics teachers must have rich, multidimensional pedagogical content knowledge.

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