The Science and Mathematics of Triangles

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A Brief History of Mathematical Sciences

Science	Mathematicised
Geometry	~ 700-300 BC
Mechanics	~ 1500-1800 AD
…	…
Probability	~ 1600-1930’s AD

Some prior knowledge of construction

- The perpendicular bisector of a line segment.
  - Abbreviation: “⊥ bisector”.

- The altitude of a triangle at a vertex. “Drop a perpendicular from vertex to opposite edge.” The edge may need to be extended.

Midpoints

- Let ABC be a triangle.
- For a vertex V, let the midpoint of the opposite edge be V'.

Vertex	Opposite edge	Midpoint
A	BC	A'
B	CA	B'
C	AB	C'

Perpendicular bisectors

- Construct the $\perp$ bisectors of all edges.

- What do you see? Make a general statement [1].

Circumcentre O (1)

- [1] The $\perp$ bisectors of any triangle meet at a point O.
- Definition: O is the *circumcentre* of ABC.
- O may not be in ABC.
- What can you say about the lengths OA, OB and OC in general? Make statement [2].

Circumcentre O (2)

- [2] In any triangle ABC, OA = OB = OC, where O is the circumcentre.

- Definition: O is the centre of the *circumcircle* of ABC.

- Construct the circumcircle.

The scientific process

- The previous 3 slides illustrates how science works.
  - First, work with concrete examples.
  - Second, observe patterns that seem general.
  - Third, formulate a general statement, or a conjecture.
  - Fourth, make new definitions, which facilitate further exploration of the examples.

- Even by skilled hands, constructions are not perfect. A line drawn on paper is not perfectly straight, and has thickness; the paper is not perfectly flat. The empirical observations can still stimulate the formulation of useful conjectures.
- It takes experience to gauge how large a tolerable error should be, beyond which the person should improve their skill.

Altitudes

- Construct the 3 altitudes. Let D be the foot of the altitude from A, so that $AD \perp BC$. Similarly, E and F are such that $BE \perp CA$, $CF \perp AB$.

- What do you see?

Orthocentre (1)

- [3] Altitudes of any triangle meet at a point H.
- Definition: H is the *orthocentre* of ABC.
- H may not be in ABC.
- What can you say about HA and OA'? Also HB and OB', HC and OC'?

Orthocentre (2)

- [4] In any triangle, HV = 2 OV’, for V = A, B, C, where O is the circumcentre and H is the orthocentre.

Medians

- Construct medians, line segments which connect V to V', for V = A, B, C.

- What do you see?

Centroid (1)

- [5] Medians of any triangle meet at a point G.
- Definition: G is the *centroid* of ABC.
- G is always in ABC.
- What can you say about GV and GV’, for V = A, B, C?

Centroid (2)

- [6] In any triangle, GV = 2 GV', for V = A, B, C, where G is the centroid.

- The centroid of a triangle of negligible thickness is its centre of gravity (or mass): the point of balance.

About O, H, G

- What do you see about O, H, G?

- What can you say about OG and HG?

Euler line

- [7] O, H, G lie on the same line, called the *Euler line*.

- [8] 2 OG = HG.

Midpoint of OH

- Construct N, midpoint of OH.
- What can you say about NA’, NB’, NC’?
- What can you say about OV and NV’, for V = A, B, C?
- What can you say about OG : GN : NH?

The point N

- [9] NA’ = NB’ = NC’. N is the circumcentre of A’B’C’.

- [10] OV = 2 NV’, for V = A, B, C.

- [11] 2:1:3

Circumcircle of A’B’C’

- What other points lie on this circle?
- Construct X, Y, Z, midpoints of HA, HB, HC.
- What can you say about X, Y, Z?
- What can you say about the line segments A’X, B’Y, C’Z?

Nine-point circle

- [10] Feet of altitudes D, E, F.
- [11] X, Y, Z lie on the circumcircle of A’B’C’.
- [12] A’X, B’Y, C’Z are diameters of the circle.
- Definition: The circle centred at N passing through A’, B’, C’, D, E, F, X, Y, Z is called the *nine-point circle* of ABC.

Geometry as science

- The constructions have the hallmark of all scientific investigations: data collection, observation, generalisation, definition.

- For a scientist, the approximate accuracy of a conjecture in numerous concrete examples suffices to hold the belief that the conjecture holds in practically all possible examples. In this case, the conjecture is called a law of nature.

Summary of mathematics

- For a mathematician, the bar is higher (infinitely so). They want to know that a conjecture holds for infinitely many examples. The only way to secure this knowledge involves three steps:
  - Delineate the relevant examples.
  - Set up axioms, statements that are assumed true of all examples.
  - Deduce the conjecture from the axioms.

Geometry as mathematics

- We are concerned with all triangles in a plane.
- We may take Euclid's axioms in *The Elements*, or more efficiently, use all the propositions in the book. Examples:
  - A point P is on the perpendicular bisector of line segment KL if and only if PK = PL.
  - The familiar facts about similar triangle: equal angles, proportional edge lengths.
- Proofs should be supported by a sketch, but not the constructions, which can be too cluttered. We will only use an acute triangle. The obtuse case can be checked on your own.

[1] Conjecture

- The $\perp$ bisectors of any triangle meet at a point O, the circumcentre.

[1] Proof

- Sketch triangle ABC, and only the ⊥ bisectors of BC and CA. Let them meet at O’.
- Since O’B = O’C and O’C = O’A, O’B = O’A.
- O’ is on the ⊥ bisector of AB. Hence O’ = O, the circumcentre.
- In proving [1], we also proved [2]: OA = OB = OC (abbreviated since context is clear).

Theory vs practice

- Since the ⊥ bisectors meet, why not just use two of them to construct O?
  - This will get a point, but we will have no idea about the accuracy. With three, we do, even though it is unsettling that they may not meet at a point. If the little triangle is quite small, we can use roughly its centre.

- It is natural to adjust the third ⊥ bisector so that it meets the first two. How to prevent this bias?

[3] Conjecture

- Altitudes of any triangle meet at a point H, the orthocentre.

[3] Proof

- Around ABC, add three congruent triangles, so that the big triangle TSR $\sim$ ABC. A is midpoint of RS, B is midpoint of TR, C is midpoint of ST.

- Since AD $\perp$ BC and BC $\parallel$ RS, AD is the $\perp$ bisector of RS. Similarly, BE and CF are the $\perp$ bisectors of TR and ST respectively. So the altitudes of ABC meet at the circumcentre of RST.

- Since ABC and TSR are similar triangles, we also get a proof of [4]: HV = 2 OV', for V = A, B, C.

[5] Conjecture

- Medians of any triangle meet at a point G, the centroid.

[5] Proof

- Draw triangle ABC and only the medians AA’ and BB’. Let them meet at G’. Connect A’ and B’.

- ABC and B’A’C are similar, so AB = 2 B’A’.

- ABG’ and A’B’G’ are similar, so AG’ = 2 A’G’ and BG’ = 2 B’G’.

- Let BB’ and CC’ meet at G’’. Similarly, we conclude BG’’ = 2 B’G’’ and CG’’ = 2 C’G’’. Since BG’ = 2 B’G’, G’ = G’’ is the centroid G. We have also proved [6]: GV = 2 GV’, for V = A, B, C.

[7] Conjecture

- In any triangle, O, H, G lie on the same line, the Euler line.

[7] Proof

- Draw triangle ABC, indicate circumcentre O and orthocentre H roughly, so that HA ≈ 2 OA'. Draw OH and AA', and let them meet at G'.

- By looking at angles, G'AH and G'A'O are similar.

- Since HA = 2 OA', we have AG' = 2 A'G', so G' = G, the centroid. We also get [8]: HG = 2 OG.

Dance of NOH

- In triangle ABC sits A'B'C', the triangle of the midpoints. N of ABC is O of A'B'C'.
- The relationship between A'B'C' and ABC is analogous to that between ABC and TSR, the big triangle used in proving [3]: H of ABC is O of TSR. Hence H of A'B'C' is O of ABC. Also, N of TSR is O of ABC.
- As we look at the decreasing sequence TSR, ABC, A'B'C', there is a point is labelled N, O, H relative to the respective triangles.
- If we sketch the relative position of O, N, H for ABC horizontally, then do the same for TSR above and A'B'C' below, and continue one more step in both directions, we can see a dance of the points as we look down layer by layer, and in the vertical direction, we see the motif NOH hanging here and there. Where is G?

Conclusion

- The constructions of various points in a triangle constitutes a scientific investigation. Exact observations are not possible, but with some skill, one can discern patterns that seem to hold generally. After sufficient experience, the scientist may formulate such general statements as “laws of nature”, or just “laws”.
- In mathematics, one seeks a more rigorous argument, a proof. If found, a statement becomes known as a theorem. The proof ultimately rests on the axioms and some rules of deduction. The reasoning is exact, so there is less doubt about a theorem than a law. However, there is a price to pay: a theorem often does not apply directly to real life. Just look at your constructions.

Albert Einstein (1921, *Geometry and Experience*)

- “But there is another reason for the high repute of mathematics: it is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.”
- “As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.”